Chen, W. et al. (Eds.) (2016). Proceedings of the 24th International Conference on Computers in Education. India: Asia-Pacific Society for Computers in Education

Mathematics Visualization for Developing the Concept of Average

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Abstract: Computer-supported cognitive tools (CT) featuring visualization are promising e-resources that can support, guide and mediate the cognitive processes of learners in mathematics. This study was conducted to examine the effectiveness of a specially designed CT with manipulative visualization scaffolds for supporting the development of the concept of average using bar charts in classroom teaching. One teacher and 28 learners from a primary school participated. Both quantitative and qualitative data were confirmative with the use of the designed CT. An effective pedagogy achieving a significant improvement over the learners' learning outcomes was designed. Both the teacher and the learners showed preference for using the CT to teach or learn the target topic, and agreed visualization of the CT aided the teaching and learning of the way to estimate the average in bar charts with the concept of "transferring the excess to the inadequate while keeping the total frequency constant".

Keywords: average, bar charts, computer-supported cognitive tool, lesson analysis, mathematics visualization, primary mathematics

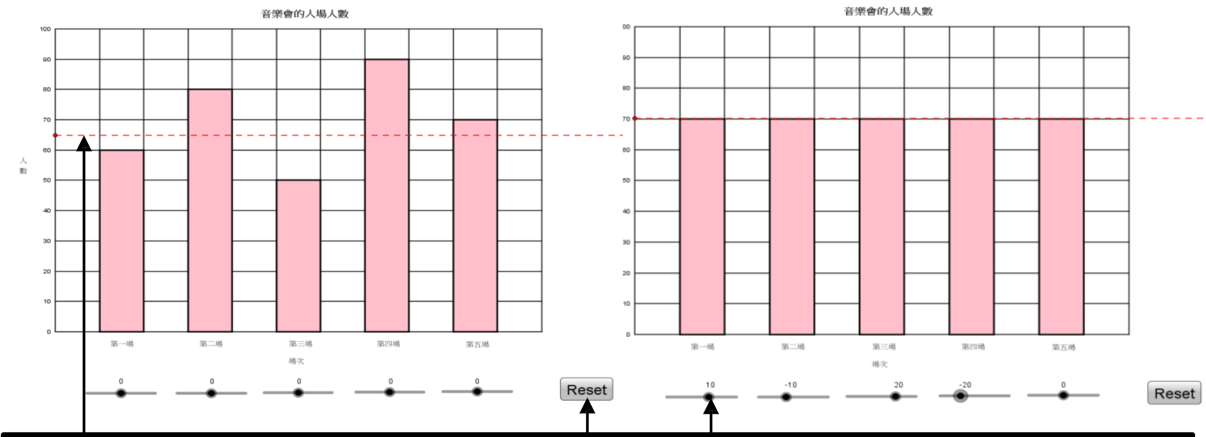
1. Introduction

Computer-supported cognitive tools (CT) featuring visual representation, graphical manipulation, and immediate feedback are promising e-resources capable of supporting, guiding and mediating the cognitive processes of learners (Kong, 2011). Mathematics visualization of CT can scaffold learners to represent the implicit mathematical objects, externalize the symbolic processing, and draw their attention to concepts or meanings underlying the mathematical objects; the manipulative feature of CT, serving as a tool of semiotic mediation, further aids outsourcing brain processing power to the computing devices. The graphical and dynamic artifacts of CT turn the mathematical abstraction into tangible and manipulative entities, and enable learners to transform the representation easily so as to explore the mathematical objects from different but interlinked perspectives. Different from "animated diagrams", CT is capable of providing immediate feedback. The interplay of the manipulation and outcome feedback from CT is in the control of learners. This interactivity may inspire the understanding on the dialectical relationship between the mathematical procedures and the related concepts.

A bar chart is a graphical display of data using bars of different heights. The capability to read, discuss and construct simple bar charts, and estimate the average from bar charts is one of the suggested teaching objectives of data handling in the primary mathematics curriculum in Hong Kong. However, the concept of average is one common learning difficulty for primary learners. Pereira-Mendoza and Mellor (1991) found that learners often made errors in making prediction from a bar chart, when the related information was not shown literally on the graph, reflecting their weakness on perceiving the broad feature of the data. The present study, therefore, was conducted in a real classroom environment to investigate the potential of exploiting the use of CT to support the development of mathematics concept of average in bar charts – a topic that emphasizes visual display and graphical manipulation.

Given that learners had no prior knowledge on calculating the average value from individual frequencies, the task requested learners to explore a non-algorithmic method for the estimation. Learners had to understand that an average means a quantity after evening out a set of quantities while keeping the total frequency or occurrence unchanged. One quick and practical strategy could be to mentally partition every bar, and to move the excess to make up the low bars. Nonetheless, this strategy could be mentally-demanding in terms of working memory for primary learners. A CT was hence created using GeoGebra to support learners to visualize the process of "transferring the excess to the inadequate while keeping the total frequency constant" for estimating the average value of bar charts.

The first design of the CT allowed learners to adjust the length of each bar by sliding the corresponding handle underneath. This manipulation may foster a dynamic process of exploration and experimentation with the data (Friel, Curcio, & Bright, 2001). Then, a movable horizontal dotted line was included in the CT to serve as a ruler to visualize that the bars have been leveled out, as well as to mark the measure of data value on the y-axis for the estimation the average value of bar charts. Another interactive feature of the CT was that a number would be shown above the handles to remind learners of the value of adjustment. When beginning learners merely adjusted the bars to be even and ignored the total frequency, they might find the numbers adding up to be non-zero. These manipulative visualization scaffolds facilitate learners to explicitly link the visual features to the mathematical meaning of average value, and so reduce the cognitive load in the paper-based task on this topic.



The height of each bar can be adjusted by dragging the slider underneath. Each unit of the bar height represented ten people. This unit cannot be further divided as designed in the CT; and so an adjustment of one unit means a change of number in ten people at a time. The number above each slider shows the value deviated from the default. Users can press the "Reset" button to restore to the default values, and adjust the location of the red-dotted line to mark the average value of the data set.

Figure 1. The cognitive tool as a visualization scaffold to support learners to estimate the average value of a data set from a bar chart "The number of audience in a concert" by adjusting the height of the bars.

The aim of this research was to examine the effectiveness of the specially designed CT in supporting the development of the concept of average on the topic of bar charts for classroom use. This research investigated the following specific research questions: (1) what are the learning outcomes and perception of learners after working with the CT? and (2) how can the pedagogical design support learners to develop the concept of average using the CT?

2. Methodology and Procedures

Participants of this study were 28 Primary 4 learners (14 boys and 14 girls) and one male mathematics teacher. Written consent from all participants was obtained before the trial teaching. The participants were able to use the computing devices before the study. They can access the CT online through a link to GeoGebra Tube with no installation of GeoGebra on the computing devices.

A mixed-methods approach was adopted. The learning outcomes and perception of learners were evaluated by formative assessment tests and questionnaire survey, respectively. A pre-test and post-test (maximum score = 47) with design in line with the revised Bloom's Taxonomy (Anderson & Krathwohl, 2001) were administered to assess the impact of the CT on learners' learning outcomes. The difficulty and length of the pre-test and post-test were matched to be identical, except the sequence and figures of the questions. All teaching objectives of the course were tested, while two of the questions assessed learners' concept of average in bar charts (maximum score = 5). A post-teaching questionnaire with a total of 69 items (Cronbach's alpha = 0.92) was administered to evaluate the learners' self-perception of the benefits of concept development on bar charts through the trial teaching. This article focused on five items concerning the concept of average. Learners rated each item on 5-point Likert scale ranging from 1 "strongly disagree" to 5 "strongly agree".

To understand the implementation and reflection of the teacher on using the CT, time allocation analysis and content analysis of the video clip of the lesson studied; and semi-structured interview were conducted. The class activities undertaken were classified into three major categories according to a classification scheme adapted from the related studies (Kong, 2011; Kong & Song, 2014), which included (a) "teaching time", (b) "student activities time", and (c) "non-teaching time". The "teaching time" category accounted for actual time spent on teacher-guided activities such as leading in-class discussions and answer-checking; and instructional activities such as lecturing and disseminating learners' work. The "student activities time" category comprised of individual and group activities with or without using technology. The "non-teaching time" covered all other class time, such as settling the class with resources distribution and solving technical problems. The content of the video clip was then analyzed according to the classroom observation protocol adapted from Copur-Gencturk's (2012) coding scheme to evaluate the teacher's performance and the degree to which learners have learnt actively, applied knowledge for hands-on problem solving, and developed understanding of complex ideas. Similar to do actual classroom observations, the project team members took notes while watching the video, discussed the notes to reach consensus. To scrutinize the teacher's reflection and justification for the use of the CT and the pedagogical design in teaching, a semi-structured interview revolving on three dimensions: (a) the outcomes of learners' mathematics learning, (b) the processes of teaching and learning in digital classrooms, and (c) the resources used for teaching and learning in trial teaching, was conducted after the trial teaching.

3. Results

3.1 The Impact of the CT on Learners' Learning Outcomes and Perception

The dependent-samples t-tests showed that the total test score increased significantly from pre-test (Cronbach's α = 0.70) to post-test (Cronbach's α = 0.72), n= 27, t=-3.78, p=0.001. Even only the questions tapping on learners' knowledge of average were considered, learners also made significant improvement, n= 27, t=-4.09, p<0.0005 (Table 1). These results suggest that the trial teaching casted positive impact on learners' learning outcomes on bar charts.

Table 1: Learners' performance on bar charts before and after the trial teaching.

Test	Mean (Maximum = 5)	*SD*	*diff*	*t*-test
Pre-test questions on average	1.59	1.95	1.78	-4.09***
Post-test questions on average	3.37	0.97		

*** $p < 0.0005$

Generally, learners appreciated using the CT to learn the topic (Table 2). They were satisfied with the interface design of the CT, for example, they indicated that they understood and felt familiar with the activities displayed by the CT, and managed to operate the CT to learn independently. They agreed that the CT provided productive scaffolds in helping them to grasp the concepts about bar charts, including the method to estimate average value.

Table 2: Learners' perception of the use of cognitive tool (CT) for learning about the concept of average in bar charts.

Evaluation item	Mean	*SD*
1. I understand the activities provided on each computer interface	4.16	0.90
2. I am confident of operating the CT independently	4.32	0.80
3. The computer interface displays are compatible with those in common learning materials	4.24	0.83
4. The CT assists me in learning how to estimate the average value in bar charts	4.20	0.96
5. The CT helps me to learn about the concepts of bar charts	4.12	0.78

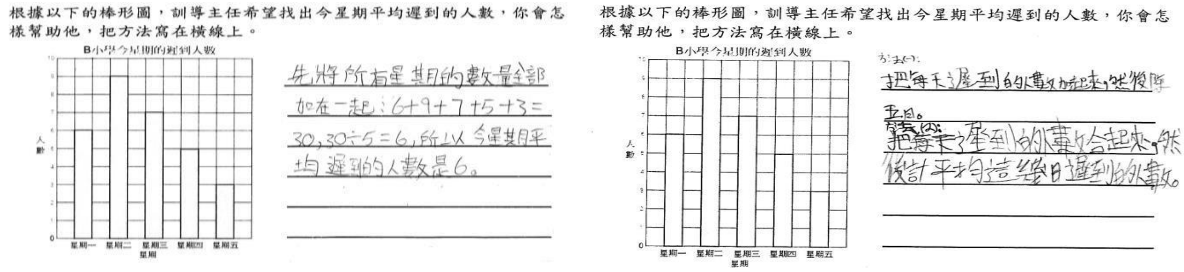
Notes. 1 "strongly disagree" to 5 "strongly agree"

3.2 Mathematics Visualization for Developing the Concept of Average Using the CT

Lasting over 35 minutes, the lesson spent 66.21% of the coding time on “student activities time”, 29.51% on “teaching time”, and 4.27% on “non-teaching time”. The content and instructional strategies of the observed lesson developed gradually into three stages and were consistent with the principles of inquiry learning in mathematics, and possessed a learner-centered structure.

3.2.1 Stage 1: Engagement and Development of the Basic Understanding of Average

The teacher first tried to engage learners to contemplate the concept of average before the class with a piece of homework on counting latecomers of a school. Surprisingly, the learners were capable of calculating the exact value or even articulating the formula for calculating the average number of latecomers (Figure 2), which out of the expected performance level. In the lesson, the teacher asked the class to share their answers and probed them to explain “what is average?” many of the learners failed to do so. According to the experience of the teacher, the discrepancy on the learners’ performance was due to the over involvement of parents or tutors in assisting or monitoring the homework process.



<u>Figure 2</u>. Samples of learner’s answer on the preparatory worksheet tapping their prior concept of average.

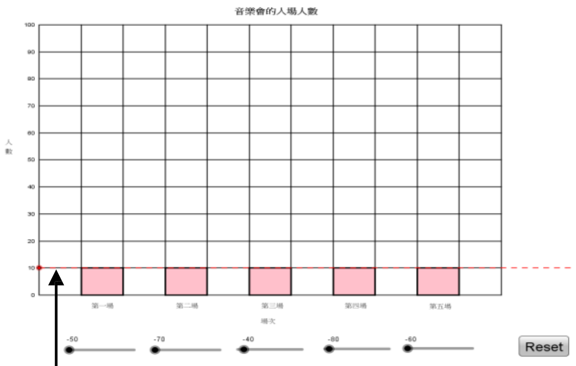
In this “teachable moment,” the teacher gave an authentic real-life problem on the concept of average by inviting learners to re-distribute chocolate beans among three learners who initially have one, two and three chocolate beans respectively. After the class consensus on the sharing outcome of two chocolate beans by taking one bean from a learner with three beans to the learner with only one bean, the teacher invited the leaners to use the CT, but no teacher guidance, to explore further the concept of average as an equal share of quantity from a set of quantities. The scaffold of the CT supported the learners to bridge the gap between the authentic real-life problem and the mathematical idea of average through the manipulation for moving the excess frequencies from the higher bars to those lower, and visualize the strategy of evening out all bars.

3.2.2 Stage 2: Learning the Principle of Maintaining the Total Frequency Constant

After the learners had explored with the CT for the concept of average as an equal share of quantity from a set of quantities, the teacher gave a challenge to the class with the use of CT to further promote learners to grasp the principle of maintaining the total frequency constant in the estimation of average in bar charts. The teacher continued the authentic real-life problem on chocolate beans sharing: he took one chocolate bean from each of the three learners, emphasized that each learner got an equal share of one chocolate bean, and asked the class to judge if this quantity of chocolate bean was the average. After the class gave the correct judge of not accepting this average value, the teacher provoked the learners to understand the need to maintain the total frequency constant by using the scaffold of the CT - to intentionally compress all the bars to the same height but of a much lower quantity in the CT (Figure 3a). The scaffold of the CT supported the teacher to easily introduce a challenge to the learners for contrasting their mathematical ideas. With the stark visual contrast from the previous bar chart presented in the CT, the class reached a concluding remark on their understanding that the total frequency of the bar chart had to be kept constant if one were to find the average from the bar chart.

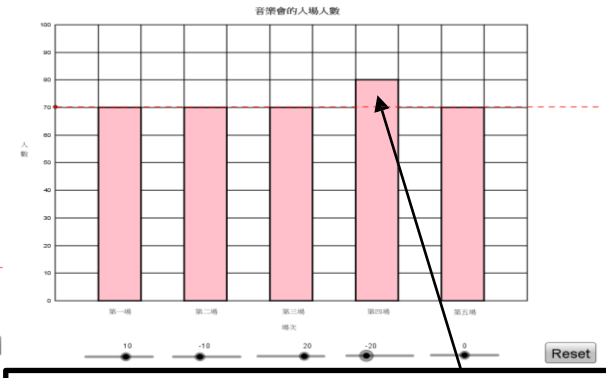
3.2.3 Stage 3: Building the Foundation of the Mathematical Procedures of Calculating Average Value

After the class had understood the concept of "transferring the excess to the inadequate while keeping the total frequency constant", the teacher made use of CT to present another challenge about a topic beyond the scope of the curriculum – the way to estimate the average from a bar chart by applying the concept of fraction. The teacher manipulated the height of a bar to make it as the highest bar remaining one unit, which represented ten people but could not be divided as designed in the CT (Figure 3b); and then encouraged learners to adapt their "old" strategies flexibly to estimate the average of the new bar chart. This challenge successfully created disequilibrium among learners, and prompted them to connect the domain knowledge of bar charts to that of fraction. The learners eventually managed to figure out a proper solution – allocating the remaining quantity of the excess bar in the concept of fraction – to disintegrate the default unit representing ten people to ten smaller units and evenly allocate two smaller units to each bar for the solution that the average of the new bar chart was 72. This task laid a foundation for learning the mathematical procedures of calculating average values in higher grade.



The teacher adjusted all the bars to be even but at a much lower value to stimulate the class to consider the total frequency of the data set in estimation of the average value.

Figure 3a. The cognitive tool as a visualization scaffold for grasping the concept of constant total frequency in estimation of average.



The teacher deliberately asked the class to find the average of a bar chart with non-integral value in order to create a cognitive disequilibrium among learners. The learners had to think flexibly to adapt their acquired strategy to tackle the task.

Figure 3b. The cognitive tool as a visualization scaffold for exploring about the procedural knowledge of calculation of average.

3.2.4 The Teacher's Reflection

The teacher was invited for an individual interview to share and reflect on his experience and performance with the pedagogical design and instructional approach of the trial teaching. The results showed that the teacher was affirmative towards the effectiveness of the CT in enhancing the teaching of the concept of average in bar charts. Precisely, he mentioned that the manipulative and interactive features of the CT allowed learners to grasp the tactic of estimating the average value. The visualization of the CT helped illustrate the abstract concept of average, facilitated the class communication, and helped learners articulate the related mathematical ideas. Nonetheless, the teacher also noticed his inadequacy on class time management, and would like to utilize the electronic communication platform more to increase the chance of idea exchange among learners in the future.

4. Discussion and Conclusion

In view of the popularity of digital technology in education, the CT was conjectured to be effective to help learners develop the conceptual and procedural knowledge of mathematics. A CT tailored for enhancing the development of the concept of average in bar charts was hence developed and examined.

The findings provided empirical evidence supporting the use of this CT in facilitating learners' development of the concept of average using bar charts. Supplemented with the carefully designed worksheets, the participating teacher managed to implement an inquiry-based and learner-centered lesson with the tailor-made CT to help learners master the concept of average through three progressive stages. In the first stage, learners learnt the basic concept of average as evening out a set of quantities; in the second stage, learners discovered that the total frequency of the data set had to be remained constant; and finally, learners were forced to accommodate their own acquired strategy of "transferring the excess to the inadequacy" and assimilate it to the concept of fraction to estimate the average value of the bar chart. The teacher's personal reflection seconded the use of the designed CT for teaching the concept of average, whilst the visualization and manipulative features of the CT were specifically mentioned in supporting learners' learning. Furthermore, learners have achieved significant improvement in the post-test over the pre-test, and favored the use of the CT.

The perceptual, dynamic and interactive interface of CT provides visualization support to conceptualize mathematical ideas (Noss et al., 2009; Rittle-Johnson & Koedinger, 2005) and experience mathematical operations (Hoyles & Noss, 2009; Moreno-Armella, Hegedus, & Kaput, 2008). Findings of this study shed light on future directions that design-based research of CT might take in other areas of mathematics. To give an instance, a CT may be designed as an interactive learning tool about how to create a bar chart. The dynamic technological environment of the CT plays the ideal role of a "hypothesis-testing" ground for learners to rescale the visual presentation of the graph. Future research direction may also go for how visualization of CT is implicated in teaching and learning of mathematics and what pedagogical support can be harnessed to capitalize on using CT for enhancing learners' mathematics concept development.

Acknowledgements

The author would like to acknowledge the Quality Education Fund of the Education Bureau of Hong Kong SAR government in supporting the research.

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