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Embodied Learning of Integer Operations Using a Multitouch Design: Touchy Pinchy Integers

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Abstract: We report the design and pilot testing of a digital application to help middle school children learn integers, *Touchy-Pinchy Integers (TPI)*. The design rationale for this touchscreen-based learning system was based on a neutralization model of integers. To design this embodied mathematics learning system, we associated commonly used touchscreen interaction gestures with mathematical meaning related to integers. Users could manipulate two collinear towers of stacked positive and negative blocks, which together represent an integer expression. The learner could add either type of block, by tapping on the appropriate part of the screen. They could also subtract a desired number of either type of block, by slicing across the blocks. They could also neutralize a pair of positive and negative blocks, by pinching in. The pinch-out actions generated a new pair of positive and negative blocks. A flipping gesture reversed the polarity of the integer expression, which corresponded to a multiplication by -1. Results from a pilot study with grade 7s students indicated that the application helped students resolve deep conceptual issues related to negative numbers, through embodied learning. We present some of our observations from the study and interpret these episodes using embodied learning theory.

Keywords: Integer Operations, Touchscreen Interactions, Tangible Gestures, Embodied Mathematics Learning, Representational Restructuration

1. Introduction

Students around the world are usually introduced to integers and negative numbers in or around middle school. This is a challenging transition for students, as the idea of negative numbers is often a novel revelation, and requires children to drastically reinterpret familiar operations such as addition and subtraction. For instance, a child may have to unlearn previously useful generalizations, such as “subtracting makes the number smaller” or “you cannot subtract a bigger number from a smaller number” and replace these ideas with a richer, more nuanced understanding of the same operations.

Typically, issues arise with the learning of integers because of misconceptions in understanding negative numbers, or operations with negative numbers. (Fuadiah et al., 2016). While some students have an intuition for integer operations, and the idea of positives and negatives canceling out (Whitacre et al., 2012), some learners develop, and carry to higher grades, fundamental misconceptions about operations with negative numbers (Makonye & Fakude, 2016). It is not surprising that children find negative numbers challenging, as past mathematicians have struggled with epistemological issues in the process of integrating negative numbers to the number system (Whitacre et al., 2012).

One proposed explanation (for why negative numbers are challenging to understand) suggests that learners find it difficult to imagine a single perfect real-world analogy, or an intuitive model that gives readily interpretable meaning to negative numbers, as well as the possible operations on them. For example, while the number line is a very pedagogically useful model, with high generality that extends up to real numbers, it does not support an intuitive explanation for what it means to subtract a negative number (Liebeck, 1990).

This difficulty could possibly be addressed using physical or virtual manipulatives or models, as these can act as concrete objects that follow predictable rules (Murray, 2018). Also, as argued by Wilensky & Papert (2010), new computational media allows radical 'restructurations' of existing ways to learn and understand difficult concepts. Mathematics educators use a class of intuitive models for integers, to help alleviate some of the issues with the subtraction of negative numbers. This approach involves establishing a conceptual metaphor that appeals to 'neutralization' or 'canceling out' of constituent positive units of an integer with their additive inverses (Flores, 2008; Murray, 2018).

Combining and extending these two model-based reasoning possibilities, as well as the representational restructuration view, we designed a digital learning application (Touchy Pinchy Integers, TPI) based on a neutralization model. The TPI system provides virtual manipulatives to learn integers. We outline some results from a preliminary study of the system with grade 7 students, who were asked to use the TPI system as a learning aid.

2. Theoretical Background

Recent work in embodied cognition and human-computer interaction has outlined a number of theoretical approaches towards the design of novel learning interfaces. We outline some of these below.

2.1 Re-imagining Human-Computer Interaction as Embodied Interaction

New computational media offers new ways of interacting with technology, overcoming some of the limitations of traditional interfaces such as keyboard-mouse input systems Touchscreens, with their ability to recognize and respond to sophisticated hand gestures such as pinching and dragging, are a good example of modern interaction technologies that engage users on a richer level, using the actions of the human body.

While touchscreens are almost ubiquitous today, more advanced interaction technology prototypes often leverage more aspects of the body to re-imagine the input-output interface. Tighter integration of the interface with the human body is a common thread in such technology designs, such as in the use of the human skin as a touch-responsive computer screen (Harrison et al., 2010). Authors such as (Wu & Hsu, 2011) have proposed methods to use human movement to help visualize large collections of information in 3 dimensional space.

One of the primary aims of developing novel interaction technology is to reduce the cognitive load experienced by the user. A possible explanation for the rapid rise of touchscreen devices is that interaction design with state-of-the-art touch-based interfaces is more intuitive and "natural" from a biological perspective, when compared to pointing and clicking using computer mice. A recent evolutionary view of Cognitive Load Theory suggests that cognitive load is significantly higher when learning tasks or information that are less optimized, in relation to the brain's evolutionary history (Paas & Sweller, 2012). From this perspective, the naturalness of common gesture-response combinations – such as drag-to-move or pinch-to-shrink – would alleviate some of the cognitive load associated with digital interactions.

Vygotsky's concept of tool-mediated cognition (Turner, 2016, p. 27–40) provides another perspective to understand the potentially profound consequences of some of the emerging technologies on human cognitive abilities. Keeping in mind Vygotsky's key assertion that tools interact bi-directionally with the human brain -- in other words our cognitive structures adapt to, and eventually mimic, the specific tools that we use to solve problems -- it is possible to argue that new computational media can significantly expand the scope of tool-mediated cognition, as it is now possible to quickly develop artifacts whose input-output relationships are not fully constrained by physical laws.

2.2 Tool incorporation and extending it to screen spaces

Recent cognitive neuroscience studies on tool-use show that active and intentional use of tools lead to changes in the neural representation of the body (the body schema). Specifically, studies with macaque monkeys (Maravita, Iriki, 2004) show that tools are ‘incorporated’ into the body schema, as the brain treats the tool as part of the body’, expanding the action space of effectors and the participant’s peripersonal space (‘actionable’ space close to the body, in this case space close to the hand). These studies have been extended to humans as well (Farne et al, 2005). This study reports that the peri-hand space after intentional tool usage expands to include not just the tip of the tool, but also includes the peri-tool-space, which includes the functional space of the tool.

Studies have shown that tool incorporation occurs even when the tool is connected to a virtual space. Macaque monkeys were trained to retrieve food by manipulating a tool and observing the hand movements through a real time video monitor, and not directly looking at the hand. The monkeys were able to use this video as a tool for food retrieval (Maravita, Iriki, 2004), and the neural body schema changed similarly to physical tool use. Further, studies have shown that human users' peripersonal space is extended onto the screen, and the cursor is treated as a tool (Gozli, Brown, 2011). In these studies, there are two worlds, the physical world and the screen world. Changes in the real world continuously cause corresponding changes on the screen, with resulting changes in the body schema and the peri-personal (actionable) space of the body. This systematic neural relationship between the body schema and virtual tool use has become integrated into our everyday lives, with regular use of computers and touch based devices.

The theoretical framework of tool incorporation has recently been used to account for the development of computational thinking in kindergarten, through learners’ active manipulation of a robotic toy (Sinha. R et al, 2023). Tool incorporation has also been used to account for the way students’ actions on physical manipulatives lead to the learning of mathematical ideas such as area (Rahaman. J et al, 2018). The incorporation account could be further extended, to account for the way abacus users generate a mental abacus, and the resulting cognitive and neural changes, such as the activation of visuo-motor areas and related gestures while verbally solving arithmetic problems (Hanakawa, T. et al, 2003).

2.3 The Role of External Representations in Cognition and Learning

Supporting the incorporation studies that show that tools have extensive cognitive effects, Kirsh (2010) has shown theoretically that external representations (ERs) – symbol-based tools used extensively during thinking, learning, and computing tasks – make cognitive contributions that are wider than optimizing internal memory load. For instance, ERs make learning processes interactive, by actively providing a stable external anchor, and also hints and prompts, which transform the reader's thoughts and ideas. This role of ERs allows users to think by altering the external representation, to arrive at a new set of insights. As ERs work as a tool that supports thinking, they transform the thinking task, by allowing building over, changing, removing or manipulating the tool (Kirsh, 2010).

Another characteristic of ERs – which also has a pedagogical benefit – is that external representations serve as a shareable object of thought. When the learner represents or works with the ERs, educators are able to observe the thinking and learning processes, which are otherwise not available. Given this structure, the educator is able to participate in the meaning-making process of the student, and intervene at the right moment to guide the learners' thinking. Abrahamson and Garcia (2016) discusses a similar idea – distributed coenactment – while examining the pedagogical outputs during the use of an embodied mathematics learning tool (The Mathematical Imagery Trainer).

Mathematics uses ERs extensively, in terms of numbers, symbols and diagrams, to capture the essence of the concepts and also lock-in the key ideas. These symbols once written down don't decay over time like mental structures. This available structure helps learners to build on their ideas. Even though ERs are symbol-based, they could be understood as functioning in ways similar to tools. It is possible that ERs are also incorporated into the body schema, as indicated by empirical studies on the cognitive and neural effects of extensive abacus use (Hanakawa, T. et al, 2003).

2.3 Embodied Learning of Mathematics

Embodied cognition theory emphasizes that the mind, body, and the environment are inextricably connected. Consequently, the body plays a significant role in thinking and acting in the world (Kosmas, Zaphiris 2018). Abrahamson and Garcia (2016) demonstrated through their instruction based embodied interaction design - Mathematical Imagery Trainer for proportion (MIT) – that learners can assimilate a mathematical concept through goal oriented hand movements. Through these actions, later reinforced using *symbolic artifacts* like cursors, grid lines and numbers on the screen, learners were able to learn the mathematical idea of proportion, making use of the movements and symbolic elements as both frames of action and reference. The symbolic artifacts allowed learners' engagement with the given tasks to be shifted closer to mathematical reasoning and visualization.

In a related thread, Sinclair (2014) argues that touch technologies allow the hand's actions to have immediate and unmediated impact, as changes in the screen are directly related to the intentional touch. She developed a novel interactive system based on multi touch - Touch counts - to develop and support basic numeracy in young children, including counting, addition and subtraction. In this system, touch media gestures – like tapping, pinching, flicking etc. – are given mathematical meanings. This structure allows learners to explore the nature of numbers, while interacting with the touch media using gestures and receiving real time feedback. Touchy feely vectors (Karnam, D et al, 2018) is another touch-based design to learn mathematics through embodied interactions. This system allows learners to learn vector concepts like addition and resolution, through the active manipulation of vectors directly using touch gestures.

Extending these theoretical ideas on embodied cognition and successful applications that help students learn mathematics in an embodied way, we have designed a system – Touchy Pinchy Integers (TPI) – to help students explore the concept of integers through embodied interaction. Apart from the embodied interaction, having TPI as a dynamic external representation reduces the cognitive load of learners, by helping them visualize the operations, and reducing their reliance on memorized rules of integer operations. TPI can be used as a tool during problem solving sessions related to integer operations. After some experience working with TPI, the system also works as an imagination tool, helping students visualize integer operations, especially when solving problems using static media, such as text and figures. This use is similar to the way experienced abacus users generate and manipulate a 'mental abacus', when trying to verbally solve arithmetic problems (Hanakawa, T. et al, 2003).

3. Methods

3.1 Learning System Design

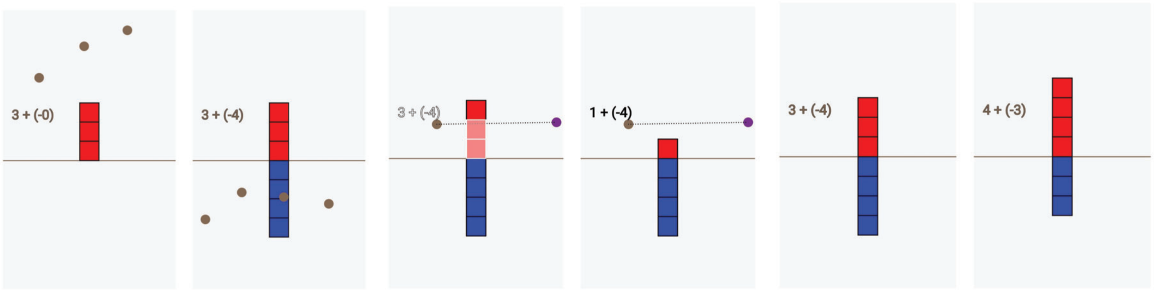
The TPI application is presented as a single screen, with a horizontal line dividing the top and bottom halves. On either side of the line, there is a positive and negative tower, each consisting of red and blue stacked blocks respectively. Each block represents a unit of the appropriate sign, and all the blocks additively represent the integer expression. Thus, 3 red and 2 blue blocks would stand in for the expression 3 + (-2).

The user can manipulate the number of blocks in each tower, using a number of touch gestures, each of which is mapped to an operation on the integer expression. Performing a gesture triggers an animation, which can result in a change of state (i.e. change in the number of blocks in either or both towers). On the screen at all times is the expression m + (-n), in which m and n represent the current number of positive and negative blocks respectively. The system is available for interaction at the following link:

https://integercountsdevelopment.melwinaalbuque1.repl.co/

The following touch gestures lead to operations on the expression:

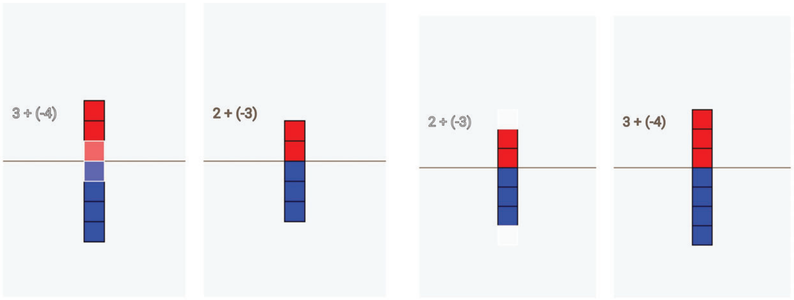
- *Tapping*: Tapping either the positive or negative side of the screen with k fingers at a time leads to addition of k blocks on the same side of the screen. The expression is incremented by +k or -k depending on the side of tapping. See Figure 1 (i) for a concrete case.
- *Slicing*: Swiping vertically across the top face of the ith positive block removes the base-most i blocks from the positive tower and subtracts i from the expression. Slicing across the top face of the ith negative block removes the base-most i blocks from the negative tower and subtracts -i from the expression. See Figure 1 (ii).
- *Flipping*: Swiping vertically across the division line causes the red and blue towers to swap colors and polarity on the screen. The overall expression is multiplied by -1. See Figure 1 (iii).



(i) (ii) (iii)

Figure 1 (i). Addition by tapping. Left: Adding +3 by tapping the top half with three fingers. Right: Adding -4 by tapping the bottom half with four fingers. Brown dots represent fingertip positions and are only for visualization - they do not appear to the user. *(ii).* Subtraction by slicing. Left: State before subtraction (subtraction animation has just begun). Right: State after neutralization. Gesture details shown for visualization (not visible to user): brown dot represents swipe start position, purple dot represents swipe end position, and dotted line represents linearized downsampled swipe trajectory used for computation. (iii)Flipping. Left: State before flipping. Right: State after flipping.

- *Pinching In*: Pinching in with one finger each on the positive and negative sides of the screen neutralizes one pair of blocks from each side - assuming there is such a pair. The overall value of the expression is conserved, but both the positive and negative terms have their magnitudes decreased by 1. See Figure 2 (i).
- *Pinching Out*: Pinching out with one finger on each on the positive and negative sides of the screen generates a one positive and one negative block. The overall value of the expression is conserved, but both the positive and negative terms have their magnitudes increased by 1. See Figure 2 (ii).



(i) (ii)

Figure 2 (i). Neutralization by pinching in. Left: State before neutralization (pinching animation has just begun). Right: State after neutralization.

(ii). Reverse Neutralization by pinching out. Left: State before reverse-neutralization (pinching animation has just begun). Right: State after reverse-neutralization.

3.2 Pilot Study Design

A pilot study using the TPI system was conducted with 5 grade 7 students in a residential school in Maharashtra, India. TPI was given to them for 15-20 minutes on a daily basis, for 1 week. The students were selected by their mathematics teacher based on their performance in their math classroom. All students were from different performance backgrounds. They were already introduced to integers and integer operations in class 6. The sessions started with a pretest, where the students were asked to solve integer-based addition and subtraction problems. At the end of the session the students were asked to explain how and why they got their answer. After the pretest, students were introduced to TPI using an ipad. They were then asked to practice the basic gestures, resulting in changes in the screen.

Following this phase, students engaged in a series of gradually advancing tasks, for 5 to 6 days. The tasks involved various operations, including: adding numbers with the same sign and opposite signs, subtracting a small number from a big number with the same sign (on day 1); subtracting an integer from zero, subtracting a larger number from a smaller one with the same sign, subtracting integers with opposite signs (on day 2); multiplying a given integer by (-1), multiplying an arithmetic expression by (-1), performing integer operations using TPI for a written expression (on day 3); exploring the triple nature of the minus sign through TPI (on day 4). In each session, the students practiced tasks from previous sessions. Day 5 was dedicated to practicing all the tasks learned using the TPI system. The students also performed integer operations for written expressions on this day. Since this was an initial study, the tasks and instruction were adjusted based on students' interactions and difficulties. For instance, the concept of screen value and multiple representations [like 2 = 3 + (-1) = 4 + (-2)] emerged during a session with one of the students.

The final stage was a post-test, where students were encouraged to perform arithmetic operations using an imaginary simulation of TPI. The questions in the post-test were similar to those in the pre-test, but this time students were immediately asked to explain their reasoning after each question. This change was made because we noticed that students had difficulty recalling their reasons when asked at the end of the test. Data collected during the study were: audio of students explaining their reasons for the integer operation (during pre-test and post-test), images of answer sheets or rough sheets created by students during pre and post test, videos of students' usage of TPI with aerial view of the software, field notes.

4. Results

After the pretest, when students were asked to support their answers with reasons, all children provided the rules they learnt in grade 6 as a reason. A few children were able to give reasons using the number line. When two negative signs were next to each other, every child instantly used the rule 'minus into minus plus', and that remained the reason for their answer. After the post-test analysis, we found that students made use of the mental image of TPI and gestures, similar to abacus experts relying on the mental image of the abacus and gestures while solving verbal problems. Among the 5, three used pinching (pinching in for addition of opposite integers) gestures, and one child used the tapping gesture, when asked to add integers of the same sign.

Two other students, who were comfortable with the rules and got all the answers right during the pre-test, used the rules again during the post-test. When asked why they chose to work and reason in terms of rules even though they worked with TPI, they mentioned that they have been practicing a long time with rules. Even while using rules to solve problems, some children used TPI to confirm their answers, using the system as an additional tool.

This allowed them to explain their answers more clearly and with greater confidence, compared to the pretest. The TPI system allowed students to avoid blindly applying rules and to prevent the misuse of rules in inappropriate situations.

5. Discussion: Embodied Learning of Integers Enabled By TPI

A number of interesting embodied learning episodes were seen during students' interactions with the TPI and later. We discuss these here, based on the theoretical frameworks outlined in section 2.

- *Thinking outside with the new representation*

External representations allow us to execute some cognitive processes outside. These processes cannot be executed 'inside', i.e. with just our minds (Kirsh, 2010). In the case of the interactive external representation created by the TPI system, the value 3 can be decomposed into three units, and split across three fingers and when tapped on the screen. This leads to three boxes summoned on the screen, as 1+1+1. This representation of individualized units allows learners to appreciate the neutralization process when a positive and negative unit come together (added). Similar to other neutralization models used in traditional mathematics classrooms (two color button model, token model, or card model), TPI also provides a visual element for the neutralization process. However, in the other models, the user has to manually remove the two coloured units for the neutralization operation. Even after removal, the units are still kept aside somewhere in the real world. But in TPI, pinching-in positive and negative units with two fingers, the two units are animated in such a way that they actually come together to neutralize and vanish into thin air, to generate the value zero. Children were surprised to see this animation of addition of opposite integers. A child noted that during this process, when the boxes on either side are not the same, one kind of box runs out. This process allowed students to observe the reason for one of the rules – to add integers with opposite signs, find the difference and put the sign of a big number. Sign of the bigger number is the same as the kind of boxes that remain on the screen after the pinching-in process. In this way, the TPI external representation, based on interactive media, allowed students to think about things that were unthinkable without the new media system (Kirsh, 2010).

Similarly, with the button or token model, the reverse neutralization operation is done by manually introducing a pair of opposite coloured units. But in TPI, pinching-out with two fingers from the central horizontal line leads to two opposite units resurrecting on either side. This illustrates the decomposition of 0 into +1 and - 1, which provides an implicit avenue to understand a rather deep idea – zero can now be seen as potentiality, rather than emptiness.

Students mentioned that TPI allowed them to think and manipulate with the boxes present outside in the system, instead of relying on rules. This allowed them to not worry about forgetting rules, and not fear the use of rules in the wrong context. Students also mentioned that while they were using the cards for integer operation, the process became messy and confusing, particularly for subtraction. TPI was easier to work with, and made the operation easy to understand.

- *Thinking clearly with varied touch screen gestures*

The TPI system assigns mathematical meanings to gestures used on touch screen. This structure allows children to explore mathematical concepts easily, by manipulating and controlling the TPI system through simple touch screen gestures. The screen split into two halves, and operations in the two halves, gave students an embodied understanding that positive units belong to the upper half and negative units belong to the lower half. Based on this embodied understanding, students moved their hands to the appropriate half of the screen quickly during tasks and explanations, while mentioning positive or negative units. This sense of direction helped reduce mistakes students make while adding integers of the same sign or different sign.

Subtraction in integers can be viewed as a change in value, movement on a number line, or taking away some units from a given unit. TPI follows the taking away idea. There are 3 different functions of minus sign when it comes to integers - unary, binary, symmetry (Vlassis, 2004). Unary defines a negative number -2, binary refers to the binary operation of subtraction 5 - 2, symmetry corresponds to the resultant additive inverse - (-2). Children seem to have internalized the concepts of unary and binary, and were able to use the appropriate gestures when they were asked to demonstrate -2 or 5-2 using TPI. They tapped 2 fingers on the lower bottom to represent the unary function, they tapped 5 and sliced off 2 boxes for the subtraction operation.

The function of symmetry required facilitators demonstrating the flipping gesture, and discussion about this operation. Students observed that flipping got them the additive inverse of the number that was flipped. With this observation, students were able to predict the answer when an expression with positive and negative numbers (units on both halves of the screen) was flipped. Students were also able to observe that on flipping both the numbers were replaced by their respective additive inverse. This allowed them to understand that although the sign '-' is the same, the function it possesses depends on the position in the expression.

The change in the expression caused by various minus signs is thus captured with the right nuances in TPI, with appropriate gestures for unary, binary and symmetric function. A question like - ((-5) - (-2)) will require the student to use 3 different gestures for the similar looking minus sign in the expression.

- *Thinking with a new concept – screen value*

The concept of screen value came up during one of the sessions as a key idea, in the context of subtraction of big numbers from small numbers and subtracting positive numbers from negative numbers or vice versa. The idea of screen value is similar to the idea of net worth in the credit-debt model of integers, where the sum of credit and debit is the net worth. Screen value is the value one would obtain on pinching-in boxes from either side till there is only one color left on the screen. For a question like 3 - 5, the student could take away 5 from a screen value of 3 i.e, take 5 from 5+(-2) which results in -2. In this process the students pinches out (reverse neutralization) to keep the screen value constant. The idea of reverse neutralization, generating a pair of +1 and -1, helped students with the operations of integers. This idea is not used in conventional classrooms for such problems. Since pinching in and pinching out are core to the TPI system, after using these operations, students start thinking in terms of a new concept – the screen value.

- *Transformation in thinking with TPI, and incorporation*

The intervention tasks had three stages, spread across 5 -6 days. Students practiced the previously learned tasks during this period, and their thinking with TPI got refined across these stages. The first stage was performing operations using the TPI system. The second stage involved predicting outcomes by mentally simulating the system, and then confirming the answers by performing the operations on the TPI system. In the third stage, during the post test, students had to mentally simulate TPI to solve the given integer operation. This was observed through students' gestures of tapping and pinching in the space in front of them, while solving the problems given in the post test. As students used gestures even when there was no system in front of them, the children were using TPI as an imagination tool for integer operation, similar to expert abacus users' use of a mental abacus. This use of TPI as an imagination tool suggests that TPI was incorporated into students' body schema, through their actions on the touch interface.

6. Redesigning TPI

Apart from the above embodied learning patterns, we also found some usage patterns that could be helpful in redesigning TPI, for better learning and usability. We outline two classes of patterns below.

- *Redesigning TPI to think mathematically*

Abrahamson and Garcia (2016) discusses how symbolic artifacts like grid and numerals allowed students to bring their action-based engagement and manipulation strategies closer to mathematical reasoning. In the TPI system, the central horizontal line allows students to use it as a frame of action for pinching-in, pinching-out and tapping, to obtain positive and negative numbers. In addition to this structure, TPI also has an expression on the screen m+ (-n), for quick reference to the number of boxes present on either side of the horizontal line. We observed students counting slowly while they are expected to subtract a value. We are planning to include grid lines, or parallel horizontal lines with corresponding integers above and below the existing central horizontal line. This structure will allow the students to identify the boxes without counting them.

- *Acclimatizing to the TPI environment*

Almost all the students initially found the pinch-out and pinch-in gestures difficult. With regular practice with the TPI system, the child was able to figure out how to make a particular gesture work with the system. The teacher could only demonstrate how it is done, but to actually make a particular gesture work, practice was needed. The practice helped students adapt the system to their organismic idiosyncrasies, such as the length of the hand, suppleness of the fingers, or the pressure and distance between the fingers the student uses over the screen (Abrahamson and Garcia, 2016).

Students also had difficulties with the slicing gestures. While subtracting 2 from 5 using the TPI system, one needs to slice off the second box from the center line. But a few students naturally moved their fingers towards the tip of the tower, and sliced the second box from the top. This gesture has a rationale: if we go according to the cartesian plane, then the second box is counted from the reference line. And irrespective of the length of towers, if 2 units have to be subtracted, users would prefer the slice to happen at the same place. By slicing the second box from the center learners preserve both these features.

Gesture usage in TPI had many other behavioral patterns. Even after repeated instructions about the possibility of using two hands for operating TPI, some students preferred to use one hand. The way students decomposed a particular number was also interesting: some students use 3+3, 4+4 for 6 and 8 while some use 5+1, 5+3 for the same.

We plan to further study these and other similar behavioral patterns, and explore their potential for designing new pedagogical elements and tasks, which could improve the use and effectiveness of TPI.

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