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Empirical Study on Errors of Mathematical Word Problems Posed by Learners

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Abstract: Problem posing by which learners create problems by themselves has been identified as an important activity in mathematics education. However, problem posing is a heavy task for both learners and teachers because it is a divergent task that has various possible answers. To develop problem posing skill of leaners, it is indispensable to evaluate posed problems, particularly when they include errors in mathematical structures. To provide a basis in designing computational support for addressing errors to improve problem posing skill, this study empirically investigated errors of mathematical word problems posed by novices. Undergraduates were engaged in a problem-posing task where they were asked to pose many, diverse and unique problems from a problem initially given. Posed problems that included errors were analyzed, with the result indicating that when the undergraduates failed to pose problems, their problems mostly had errors regarding setting constraints. We then discussed how to approach errors in problem posing by computational systems.

Keywords: Problem posing, mathematical learning, word problems, computational support

1. Introduction

In addition to solving problems given by a teacher or textbook, problem posing, by which learners create problems by themselves, has also been identified as an important activity in mathematics education. In fact, some mathematicians and mathematics educators have pointed out that problem posing lies at the heart of mathematical activity (e.g., Polya, 1945; Silver, 1994). Problem posing is necessary skill in problem solving of everyday life. Because structured problems are not provided when using mathematics in everyday life, problem solvers must recognize and formulate problems by themselves (Singer, & Voica, 2013). Nevertheless, learning of problem posing is hardly adopted in school classrooms. One of the reasons for this may be that problem posing imposes high loads on both learners and teachers. Problem posing is a divergent task that requires novel idea generation from learners. Therefore, it is a heavier task for learners than problem solving. It is also heavy for a teacher. Because problem posing does not have a unique answer but various possible answers, a teacher must evaluate each of problems posed by learners and respectively provide feedback. To develop problem-posing skill feasible in everyday life, it is indispensable to individually evaluate posed problems particularly when they include errors in mathematical structures. Novice learners have difficulty in composing structures of problems, and they can fail in it (Kojima, Miwa, & Matsui, 2010a; 2011b). Therefore, incorrect problems including errors must be addressed to improve learner skill. However, it is in general a difficult task to evaluate incorrect responses in a divergent task.

Several studies have addressed evaluations of posed problems by developing computational support systems. Some of them adopted peer evaluations among learners (e.g., Barak, Rafaeli, 2004; Takagi, Teshigawara, 2006; Hirai, Hazeyama, & Inoue, 2009; Yu, Liu, & Chan, 2005). They basically use multiple-choice format problems that question declarative knowledge. They have not adapted to domains of problems that have structural features, such as mathematics. Hirashima and his colleagues implemented learning environments for learning by problem posing in arithmetic word problems and physics problems (Hirashima, Yokoyama, Okamoto, & Takeuchi, 2007; Yamamoto, Waki, & Hirashima, 2010). Learners pose problems by combining cards of sentences or physical objects initially provided in these environments. Because the range of problems possible to pose is limited in the environments, they can automatically evaluates posed problems and provide feedback for errors. However, these studies focus on improving learner understanding of domain knowledge or problem-solving skill through problem posing. There have not been sufficient studies regarding support for improving problem-posing skill in terms of errors.

To provide a basis in designing computational support for educating problem-posing skill, this study empirically investigated errors of mathematical word problems posed by novices. Although it has been reported that novices can pose unsolvable or incorrect problems (Kojima et al., 2010a; 2011b; Leung, & Silver, 1997), precise analysis of such problems has not been performed. We analyzed problems from data empirically obtained in our previous studies (Kojima, Miwa, & Matsui, 2010a; 2010b; 2011a), which includes errors (e.g., inconsistency between problem texts and solutions, or mathematically incorrect relationships). We then discussed how to approach errors in problem posing by computational systems.

2. Method

2.1 Experimental Procedures

We collected problems posed by general undergraduates of a wide range of background (e.g., psychology, computer science or welfare) in four cognitive science classes held in from 2009 to 2012. The topic of the classes was creativity. The undergraduates were engaged in a problem-posing task where they were asked to pose from a problem initially given and to write their texts and solutions on provided sheets in 20 minutes. The initial problem was the following word problem solved with a unitary equation (a single linear equationcontradictory), which is used in middle school mathematics education.

I want to buy a certain number of boxes of cookies. If I buy some 110 yen boxes of cookies, then I have 50 yen left. If I buy some 120 yen boxes of chocolate cookies, then I need 20 yen more. How many boxes do I want?
Solution.
Let x denote the number of boxes.

$$110x + 50 = 120x - 20$$

According to the equation above, $x = 7$.

The undergraduates were asked to pose as many and different problems as possible. They were encouraged to pose diverse problems different from the initial problem, and unique problems different from those posed by the other undergradutes. They were also instructed that their problems should be solved with unitary equations and middle school students should be able to solve it.

In three of the four classes, undergraduates learned a problem as an example of output of problem posing in the domain of word problems solved with simultaneous equations before start of the task. In the class at 2010, undergraduates learned the example by solving it. In 2011, undergraduates reproduced the same problem as the example. In 2012, undergraduates evaluated the example in terms of the originality and usefulness. The purpose of the previous studies in 2010, 2011 and 2012 was to examine whether learning of the example had impact on the problem-posing task. However, we do not discuss this point because the purpose of this study is to examine errors that novices make in the problem-posing task.

2.2 Data and Analysis

Some of problems posed by the undergraduates included errors. These error problems were excluded from analysis in the previous studies because they had no answers, had answers that were arbitral values, or had solutions that were inconsistent with their texts.

We classified the errors into the following categories according to their matters, expanding classifications by Leung and Silver (1997).

- *No mathematical relationships*: Problems of this category had texts that included numeric parameters, but they included no mathematical relationships among the parameters.

- *Inappropriate relationships*: This category had problem texts that embedded relationships among numeric parameters, but they were mathematically inappropriate.
- *Inconsistent solutions*: This category had problem texts that had mathematically appropriate relationships, but solutions that undergraduate described were not consistent with the relationships in the texts.
- *Contradictory constraints*: This category had no answers because constraints in problem texts were contradictory.
- *Insufficient constraints*: This category had answers of arbitral values because problem texts did not provide constraints enough to lead unique answers.
- *Excessive constraints*: This category had problem texts that had mathematically appropriate relationships, but the problems can be solved without formulating equations from the relationships because of excessive information.

3. Results

Five hundred and forty seven undergraduates participated in the problem-posing task in the four classes. They posed 854 problems. Eighty two of the posed problems were not in the domain of word problems solved with unitary equations. Forty two of the other 772 problems included errors. Examples of error problems in each category are as follows.

- *No mathematical relationships*

On a school trip, teachers distributed lunch boxes of beef or chicken to students. Ten teachers distributed lunch boxes of beef and 5 teachers distributed lunch boxes of chicken. Three students were waiting to receive beef, and 2 students were waiting to receive waiting to chicken. How many students were there?

The solution of this problem was not described. This problem includes no mathematical relationships that can find the number of students.

- *Inappropriate relationships*

An express train is 4 times faster than a local train. Today, the train service was delayed due to an accident. An express train arrived at the terminal station 40 minutes later than usual, and a local train arrived at the terminal station 10 minutes later than usual. Find minutes it usually takes for an express train to arrive at the terminal station. The delay time of an express train was the same as that of a local train.

Solution.

Let x denote minutes to arrive the terminal station.

$x + 40 = 4x + 10$

According to the equation above, $x = 10$.

This problem does not pose appropriate information to formulate the solution described. It should pose a setting such as "a local train leaving in 10 minutes and an express train leaving in 40 minutes will arrive at the terminal station at the same time."

- *Illegal constraints*

I want to buy a certain number of writing materials. The amount of 3 pencils and a 120 yen notebook is equal to the amount of a red pencil and the same notebook. A red pencil is 20 yen more expensive than 3 pencils. How much a pencil is?

Solution.

$3x + 120 = (3x + 20) + 120$

According to the equation above, $x = 40$.

This problem has no answer because the right and left sides of the equation are not equal. A notebook whose price is different should be bought with a red pencil.

- *Inconsistent solutions*

A teacher is planning to divide students into a certain number of groups. If 3 students are assigned to each group, then 2 students are left. If 4 students are assigned to each group, then 1 student is left. How many students are there?

Solution.

Let x denote the number of students.

$x / 3 + 2 = x / 4 + 1$

According to the equation above, $x = 17$.

Although x is not 17 but -12 in this equation, that is not the critical matter. The solution described is not correct. The correct solution of this problem is "$(x – 2) / 3 = (x – 1) / 4$" and the answer is 5.

- *Insufficient constraints*

I want to buy a certain number of boxes of cookies. If I buy x 180 yen boxes of cookies, then I have 100 yen left. If I buy x+1 180 yen boxes of cookies, I need 80 yen more. Find the value of x.
Solution.

$$180x + 100 = 180(x + 1) - 80$$

According to the equation above, $x = 5$.

This equation is changed to "0 = 0." The answer of this problem is any natural number. Information for the left and right sides of the equation should be different.

- *Excessive constraints*

I am in a book store and I have 900 yen now. If I buy 2 books, then I have 100 yen left. If I buy 3 books, then I need 300 yen more. How much does the book cost?
Solution.

$$2x + 100 = 3x - 300$$

$x = 400$.

This problem can be solved with "(900 – 100) / 2" or "(900 + 300) / 3" without formulating the equation. The parameter "900 yen" should be removed.

Figure 1 indicates the proportions of error problems in each category. About 75 % of the errors were due to matters in setting constraints posed in problem texts (contradictory constraints, insufficient constraints or excessive constraints).

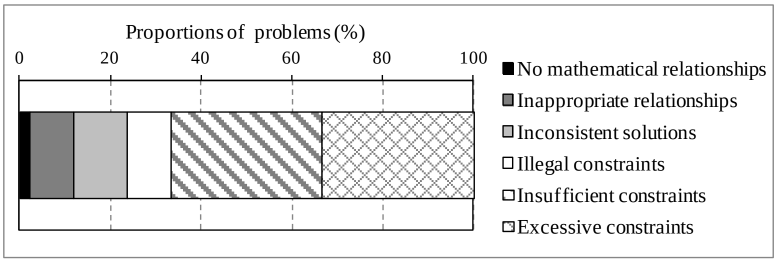

Figure 1. Proportions of error problems in each category

4. Discussion

The result shown in the previous section revealed that when the undergraduates failed to pose problems, their problems mostly had errors regarding setting constraints. They also indicated that the undergraduates did not necessarily find they made errors. Most of the error problems described their solutions and answers as shown in the examples above, even though the solutions were incorrect.

The undergraduates did not pose difficult problems, but rather posed simple problems. The initial problem given in the problem posing task is an elementary problem used in middle school mathematics education, which must be quite easy for undergraduates. In 730 posed problems other than the 42 error problems, 359 (49.2 %) had the same solution structure to the initial problem. The other 371 problems (50.8 %) had different solution structures. In terms of the complexity of solution structures (the numbers of mathematical operations needed in solving problems), 170 (23.3 %) of the 371 problems of different structures were more complex than the initial problems. Therefore, 76.7 % of the 730 posed problems were as simple as the initial problems, or simpler than it. Most of the error problems were also simple. As the example of excessive constraints, some of them were supposed to have the same solution structure to the initial problem. Although the number of the error problems was few, the undergraduates posed simple problems in the domain whose target learners are middle school students. If middle school students pose problems in the domain, they would pose more error problems. Accordingly, education of problem-posing skill must generally need support for detecting or correcting errors because many errors are expected.

In the cases of contradictory constraints or insufficient constraints shown above, errors can be detected by solving equations in solutions. It may be possible to prevent excessive constraints by bringing attention to a numeric parameter in a problem text when the parameter does not appear in it solution. However, it is indispensable to analyze problem texts in terms of semantic structures in order to accurately check errors or individually provide feedback. Such analysis is impossible for current computational systems due to technical limitations of natural language processing. Therefore, support by computational systems should aid novice learners in checking and correcting their problems by themselves. One approach to aid check by learners is to present them with an example of an error problem and have them verify whether their problems include the same error. Learning of errors in problem posing may enable learners to improve their problem-posing skill. Thus, we are planning to examine the effect of verifying an error example by learners and implement a system that supports verification of error examples in the future work. When an error problem of contradictory, insufficient or excessive constraints is posed, a computational system can present an example similar to the problem by detecting its error type and parsing a structure of its equation. Such a similar example is considered to be useful for a learner in finding and correcting an error of his/her problem. However, we have to devise a new method to provide appropriate examples when problems of the other errors are posed.

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