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Interactive Vectors For Model-based Reasoning

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Abstract: Reasoning about the structure and behavior of physical phenomena using abstract and concrete models (model-based reasoning, MBR) is a key thinking skill in science and engineering practice. One of the key areas MBR is introduced in the curriculum, particularly the use of abstract models, is applications of trigonometry, such as calculating heights and distances. In India, high school and pre-college (9-12 Grades) trigonometry curricula include three broad MBR cases: i) heights and distances (ratios in right triangles), ii) resolution and addition of vector quantities (projections in a unit circle to give the rectangular components), and iii) periodic systems (represented as sinusoidal functions). Students find trigonometry and its MBR applications difficult to understand, possibly because reasoning in this domain requires handling cognitive (internal/abstract) operations and symbolic (external/concrete) operations simultaneously, in different and complex ways, across these three MBR cases. A particular source of difficulty is the relationships between these trigonometric operations, which are not clear across the three cases. We are developing an interactive new media system to help students learn model-based reasoning, based on MBR applications of trigonometry. Here we focus on vector resolution and addition, a key application supporting MBR. In the existing curricula, trigonometric and other concepts related to vectors are scattered across 4 textbooks, and students find it hard to integrate these scattered concepts. We report a study outlining how the new media tool helped students integrate the concepts involved in vectors, and the insights from the study for redesign, particularly to support MBR.

Keywords: model-based reasoning, vectors, trigonometry, embodied cognition, new media

1. Introduction

Reasoning using models is a central thinking skill in science and engineering. According to Gilbert (2004), models “function as a bridge between scientific theory and the world-as-experienced (‘reality’). They can be simplified depictions of a reality-as-observed, produced for specific purposes, to which the abstractions of theory are then applied. They can also be idealisations of a possible reality, based on the abstractions of theory, produced so that comparisons with reality-as-observed can be made". Trigonometry is one of the key mathematical topics where students learn model-based reasoning (MBR) in the later high school curricula. It is also a topic with wide applications in advanced mathematics, physics and engineering. Research studies (Gur, 2009; Jackson, 1910; Orhun, 2004; Yusha’u, 2013) report that teachers and students find trigonometry a hard concept to teach and learn. Byers (2010) reports a detailed study of trigonometric representations in the Canadian curricula, particularly in the transition from the secondary school to college mathematics, and points out many potential sources of difficulty for students. Gur (2009) also identifies these problem areas, and suggests that the difficulties come from the complex nature of trigonometric symbols.

In India, trigonometry is introduced to students during the later parts of high school, as part of Mathematics, along with many applications across Physics. There are three kinds of model-based reasoning applications of trigonometry in the higher secondary curricula in Indian Schools. These cases which follow the levels of understanding in geometry proposed by Van Hiele (1986) are: (i) Heights and distances: Here a real world scenario is modelled by a right triangle. Trigonometric ratios are used to link the angles (of observation) and the lengths of the triangle (heights and distances). This is one of the first applications of trigonometry, introduced after the basic definitions. (ii) Resolution and Addition of Vectors: Diverse applications in physics are modelled using vector operations, ranging from resolution and addition to products of vectors. In 11th and 12th grade physics, various physical quantities are introduced as vectors (displacement, velocity, acceleration, force, momentum, angular velocity etc.). Vector operations involving trigonometry are used to solve problems such as finding resultant forces, conserving momenta, and the effect of a set of torques. (iii) Sinusoidal Systems: Trigonometry is used to model phenomena with periodically changing physical quantities. In physical systems (like a spring mass system or a simple pendulum), chemical and electrical systems, periodically varying physical quantities can be modelled using a sinusoidal curve. Here the notion of trigonometric ratios as functions of angles is used.

Here we focus on the case of model-based reasoning using vector resolution and addition. Byers (2010) suggests that students find vectors difficult due to unfamiliarity. Difficulty in handling vectors leads to problems in handling Newtonian Dynamics (White, 1983). Students have trouble understanding the wide range of modeling applications of vector concepts, as it is difficult to follow how the vector concepts model the various real world cases. Resolution and addition of vectors thus provides a rich context to explore students' understanding of model-based reasoning using trigonometry, its applications in physics, and how new interactive media could address learning problems in this domain.

2. Textbook Analysis

We first analysed the textbooks in one of the provinces in India (Maharashtra), to understand the manner in which vector concepts are covered. Since the topics related to vectors are spread across mathematics and physics textbooks (grades 9-11), we were interested in documenting the missing conceptual links, both within a text book and between text books.

Figure 1 shows a concept map of how topics are covered and applied in the physics curriculum. Addition of vectors is introduced geometrically using the Triangle Law and Parallelogram Law of vector addition. The Triangle law is just stated, and no connection is made to properties of vectors. The conceptual gap thus gets carried over to the Parallelogram and Polygon Laws, which are proven based on the Triangle Law. Addition of vectors is thus not properly scaffolded.

Different representations of the same vector as a geometrical entity (an arrow mark ↗ with magnitude and direction) and as an algebraic entity (the rectangular components form) are interrelated using the operation called Resolution of vectors. The textbook does not clarify how both these representations denote the same vector. This could lead to students treating a vector under geometrical and algebraic descriptions differently, without a unified perspective. Similarly adding two vectors geometrically (using triangle and parallelogram laws) and algebraically (adding rectangular components) may be perceived as two entirely different operations, and hence need scaffolding to understand how they lead to the same resultant vector. The text book does not provide an integrated understanding of geometric and algebraic representation of a vector.

Further, textbooks don’t emphasize the notion of Resolution as an inverse operation of addition (adding the component vectors back will give the initial vector which was resolved). This leads to a weak understanding of the nature of these operations, and difficulty in understanding the nature of vectors and components in situations such as a changed frame of reference (like a rotated frame in an inclined plane) and also the possibility of non rectangular components of vectors. The conceptual issues here will be carried over to all the connected chapters (right block in Figure 1, various chapters in *mechanics* as well as *electricity and magnetism*).

A central finding from this analysis was that a key transition in learning vectors -- understanding the translation between the geometric mode and the algebraic mode -- is not well scaffolded. The role of trigonometric ratios, which are employed in this transition, is also not discussed. Given the way the chapters are sequenced in the physics and the math curriculum, students have little support to understand and master the application of trigonometry in the context of vectors.

An analysis of the Mathematics text books in the previous grades (blocks to the left in Figure 1) showed that trigonometry is introduced first in Grade 9. Till Grade 10, the text covers the basic definitions, and applications to the problem of calculating heights and distances. A brief mention of trigonometric ratios with varying angles is made in Grade 10. However, these connections are not emphasized enough, for the student to apply trigonometry in the context of resolution of vectors when they move to Grade 11. The mathematics textbook for Grade 11 discusses trigonometric functions, but with no direct applications in the context of vectors.

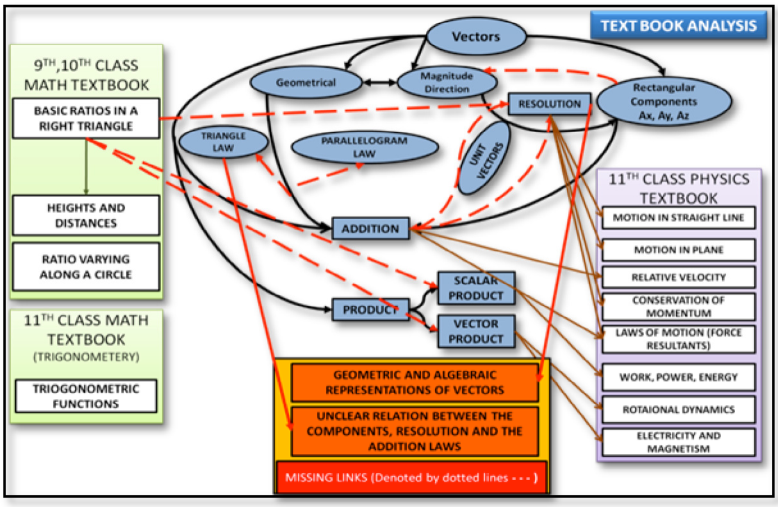

Figure 1. Vector concepts covered in Physics and Mathematics Text books (Maharashtra State). Dotted lines represent improper scaffolding, which could lead to potential conceptual gaps.

The current state of the textbooks and curricula leave the students without enough scaffolding to understand the vector concepts. These improperly understood concepts, when carried ahead to further abstract topics, leave the students highly prone to conceptual confusions, and an inability to make sense of symbolic manipulations and their roles. The only way out for them is to rote learn or devise tricks like Fatima rules (Aikenhead & Jegede, 1999; Larson, 1995).

To check if issues identified by the textbook analysis resonated with teachers, we did a round of interviews with 3 grade 11 physics teachers and 1 mathematics teacher. The focus was problems they found when teaching, and also gaps in student understanding, in the context of vectors and its applications. The interviews were semi-structured, and the teachers used paper to explain when needed. All discussions were recorded for further analysis. The inputs from teachers varied with teaching experience. For example, all teachers noted students' difficulty in resolution and addition of vectors, and finding dot and cross products. But the reasons cited by the teachers varied from procedural aspects (inability to solve a determinant to find cross product; confusions about trigonometric ratios in vector resolution; using the formulae for dot and cross products) to conceptual aspects (not understanding the notion of direction of a vector, and the geometrical addition in a triangle giving the resultant).

3. Design of the Tool

We have developed a new media intervention to address the concept integration issues identified by the textbook analysis and teacher interviews. Computational media is a new way to introduce students to complex mathematical content (Kaput and Roschelle 2013; Kaput 1992; Kaput et al. 2002; Hoyles and Noss 2003). Systems with embodied interactions have been recently found useful in learning abstract mathematical concepts (Sinclair & Heyd-Metzuyanim, 2014). These studies support the idea that mathematical cognition, and symbolic cognition in general, could be understood as embodied cognition (Ottmar, Weitnauer, Landy, & Goldstone, 2015; Landy et al., 2014). Our design approach is inspired by these developments in the cognitive and learning sciences. New media systems to learn trigonometry are not new (Lotfi and Mafi , 2012; Zengin et al. 2012), but existing systems deal with very preliminary concepts, and the design and studies are not based on ideas from cognitive and learning sciences. Our approach is based on learning sciences, identifying key gaps in the curricula, particularly in integration of concepts. The design of the tool is based on embodied cognition theory, where we seek to develop a system that allows students to learn vectors, and more broadly MBR, in an integrated fashion using embodied interaction.

The current prototype is built using JavaScript, and supports learning of vector resolution, addition (dominantly triangle law) and understanding vector components. A demo video of the computer based tool is provided as supplementary material. The tool as in Figure 2 (left), allows the user to create a vector by clicking or tapping the screen in the larger left panel, and change its magnitude and direction by manipulating the vector. The learner can view right triangle projections and the rectangular components. Side panels always show the right triangle projected on the x-axis, and the circles of all the vectors on the screen. Two vectors can be added to see the resultant as in Figure 2 (right) using the triangle law of vector addition. The changes in the magnitude and the direction of any of the component vectors results in a corresponding change in the resultant vector.

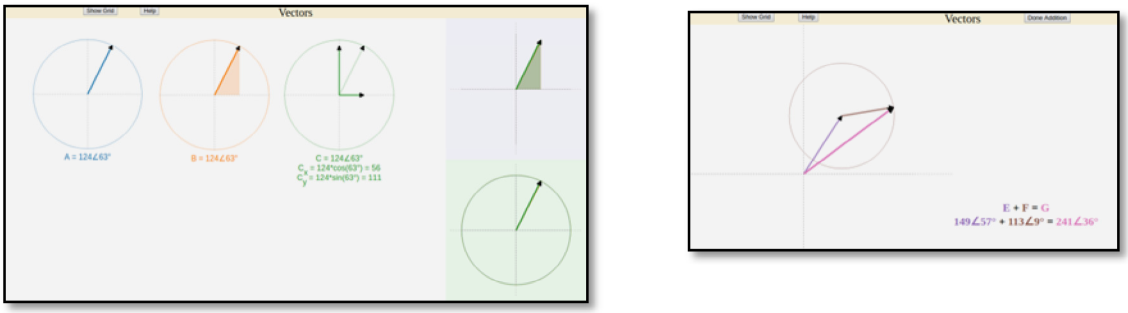

Figure 2. Snapshot of the intervention tool (left): Addition mode (right)

Simultaneous presence of components and the geometric changes modelled using the unit circle (vector inscribed in a circle), could scaffold the gap identified in the Textbook analysis. This feature also allows dynamic real time embodied interaction with vector elements, allowing learning of direction and magnitude, and also understanding how trigonometry is related to the components of vectors. The system thus allows learning both the nature of vectors and that of operations such as resolution and addition, in relation to their components. The possibility of numerous components and notion of non-rectangular components can be modelled using this tool.

4. The Study, Data Collection and Analysis Framework

A group of grade 11 students (n=49), who had finished their academic year dealing with vectors in physics, were first given a short written pre-test with 9 questions (to test pre-requisites and resolution and additions of vectors, as well as components and connections with trigonometry). From this group, 8 students were chosen (representing the range of good and bad performance) based on their responses in the pretest, and ability to externalise their understanding using text or diagrams. This group (n=8) was interviewed in the context of their pretest responses, to get a better understanding of their existing understanding.

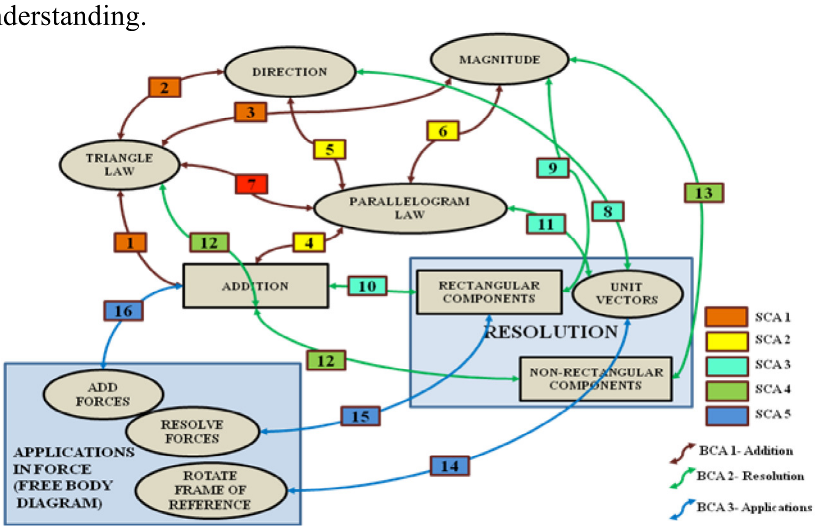

Figure 3. Concepts and the linkages

From this group, 6 students attended the intervention sessions, which involved performing tasks on the interactive vector system for about 70-90 minutes. These tasks were designed to make the students explore various features in the tool. Sample tasks were described in a video in the supplementary material. Their actions were recorded using video, written scripts (rough work), screen capture, and eye tracking (Tobii X2-60). After about a week, these 6 students were given a post test similar to the pretest (without prerequisite questions) followed by an interview in the context of their test responses.

Figure 3 shows the topics probed, categorised into three Broad Concept Areas (BCAs), which are further categorised into 5 sub concept areas (SCAs). These concept areas constitute 16 links between concepts (CLs). The pre and post test answers of all the 6 students were each analysed by 3 raters, and ratings were given to all relevant CL-question pairs. A 5-point rating scale for conceptual understanding (1 = no indication, 5 = strong indication) was developed for each of these concept links. This structure is based on studies examining the shift from conventional problem solving approach (prescribed in the text book) to a more conceptually sound explanation and judgments (Niemi 1996; Besterfield-Sacre et al. 2004; Gerace et al. 2001). The scale does not measure the correctness of the response, but rates the conceptual clarity of that particular concept link, as expressed in the answer to a given question.

If 2 or more raters found a concept link-question pair irrelevant, that pair was deemed irrelevant, irrespective of the third rater's rating. Inter-rater reliability was estimated using a weighted proportion of agreement ('2/3' for two raters agreeing, and '1' for three raters agreeing). If only one rater found a pair irrelevant, the agreed rating of the other two raters was used. The final score for each concept link-question pair is taken as the mode of the three ratings for the cases with agreement. For cases where all the ratings varied, further discussions led to 2 raters coming to a consensus. This exercise ensured an inter rater reliability of more than 67% across all the raters and the CL-question pairs. The ratings (converted to percentages) denote the strength of the CLs. This provides a comprehensive picture of the strengthening of specific CLs after interacting with the system.

5. Results and Discussion

Figure 4 (bar graph) shows the proportion of students whose understanding improved across each of the 5 SCAs (SCA1- Triangle Law; SCA2- Parallelogram Law, SCA3-Rectangular components, SCA4-Non rectangular components, SCA5-Application in context of forces). The colored lines capture the change in conceptual understanding (from pretest to posttest) of each student. The slope of each line captures the growth achieved by the student in the understanding of the CLs in that particular SCA.

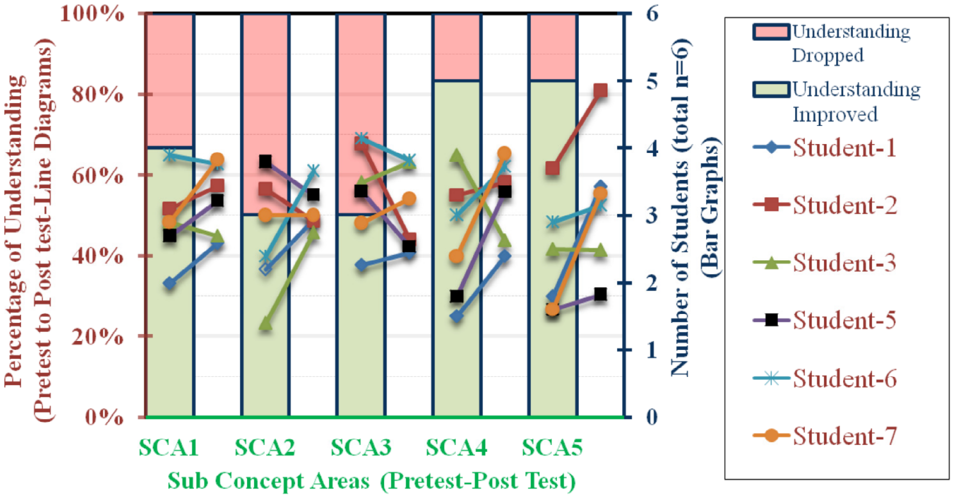

Figure 4. Conceptual understanding growth trajectories across Sub Concept Areas

Students improved in all SCAs, but in different ways. SCA1, SCA4, SCA5 show 3 students improving in their conceptual understanding. Four students improved in SCA1 (triangle law), which is expected, as the interactive system is predominantly based on triangle law. The parallelogram law (SCA2) is not very explicitly expressed in the system, but 3 students improved in this SCA. The two drops were about 2-3%. S6's drop in performance could be attributed to disruption in concept areas SCA3 and SCA4, related to components and addition. Surprisingly, only 3 students improved in the rectangular component (SCA3), even though rectangular components were part of the system. The improvement for this SCA was not more than 5-6%, this suggests this aspect of the tool needs to be redesigned. All the students whose performance dropped had pretest percentages around or more than 60%. This suggests that the system disrupted their existing concepts. SCA3 and SCA4 are closely related to components of a vector, and 5 students showed conceptual growth in SCA4, which pertains to non-rectangular components. Interestingly, all 3 students with weakened conceptual understanding in SCA3 (S2, S5, S6) have shown growth in SCA4. SCA5 (applying vectors and vector operations in the context of forces) improved in all students. Qualitative analysis of interviews showed comments supporting the above observations.

The above data suggests that the interaction with the system helped students improve their understanding of vectors in two ways: 1) understanding of triangle law and the non-rectangular components, and 2) the related disruption in their understanding of rectangular components. The students' interaction process is currently being analyzed for further insight to redesign the system.

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