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Assessing Students' Mathematical Misconceptions through Concept Maps and Online Discussion Transcripts: Inner Product Spaces

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Abstract: Knowing students' mathematical misconceptions can help educators trace the sources of the cause and help students learn conceptual and procedural knowledge in balance. This study aims to investigate students' misconceptions through students' concept maps and online discussion transcripts. This case study takes the inner product spaces as the learning subjects. The types of misconceptions are investigated based on the source of errors. Based on the findings, the author identifies students' learning difficulties and proposes certain learning activities to help students learn.

Keywords: concept map, online discussion, mathematical misconception

1. Introduction

Ball, Hill, and Bass (2005) define the word 'concept' "….as a knowledge structure of common characteristics of different substances and events captured by the human brain." In mathematics, conceptual knowledge is described as mathematical concepts and relations among each other (Baykul, 1999). Whereas, procedural knowledge defines symbols, rules, and knowledge used to solve mathematical problems. Educators agree that understanding misconceptions in mathematics are essential for teachers to help students rectify and correct them (Baykul, 1999). Meaningful and permanent learning can be possible, provided that procedural knowledge and conceptual knowledge be learned in balance (Noss and Baki, 1998). If a student possesses both conceptual and procedural knowledge, his conceptual knowledge guides him to solve by establishing a connection between basic concepts. Lee (2006) claims that one common error made by students is that they possess a procedural knowledge that is not backed by any conceptual understanding. He identifies pupils' learning difficulties in learning Algebra, such as they are not familiar with the syntax of algebra, they are confused over notations, and they find Algebra too abstract. Students' misconceptions reflect their learning difficulties.

Understanding misconceptions is essential for educators to help students overcome their learning difficulties. It is also important to highlight that errors and misconception are different; being that an error might be a result of a misconception. Migon, J (2007), in the CIAEM (1987), stated that an error takes place when a person chooses what is false as the truth; Spooner (2002) argues that misconception is the result of a lack of understanding or misapplication of a rule or mathematical generalization (as cited in Mohyuddin & Khalil, 2016). Ojose (2015) defines misconceptions as misunderstanding and misinterpretation based on an incorrect meaning. Sarwadi and Shahrill (2014) argue that, sometimes, students' errors are systematic and can casually be determined. Systematic errors often indicate misconception (Sarwadi & Shahrill, 2014). Errors are of such various types that difficult to classify accurately (Mohyuddin & Khalil, 2016). The types of misconceptions include preconceived notions, non-scientific beliefs, conceptual misunderstanding, and vernacular misconceptions. These misconceptions arise from everyday experiences, falsities learned at early ages, methods of teaching, and the use of words or notations (National Research Council, 1997). In his study, Stavrou (2014) reported recurring errors: cyclic proving, using specific examples to prove general statements, not proving conditions in a biconditional statement, and misusing definitions.

Misconceptions are exhibited in various activities, such as discussions (both face-to-face and online) and students' responses on tests or assignments, in forms of speech, text, or concept maps. Hirashima et al., (2011) propose a framework of the Kit-Build Concept Map where a learner's concept can be diagnosed automatically by comparing it with the subject experts' map. Misconceptions can also be detected through the discussion transcripts. Online discussions facilitate knowledge construction, promote deep learning, and provide output where the learners' responses can be reread and analyzed (Novicki, 2013).

Lee (2006) emphasizes that knowing the conceptual difficulties of students is helpful in planning instructional strategies to facilitate learning and to help learners overcome their learning difficulties. Hirashima et al., (2011) suggest that it is necessary to help the learners to identify and correct the errors since it is often hard for them to be aware of the inaccuracy and incompleteness.

This study investigates students' misconceptions exhibited in their concept maps and online discussion transcripts. The study takes a case of inner product spaces covered in Linear Algebra course. A previous study shows that some students make mistakes and construct the wrong generalization about the subjects (Junus, 2017).

2. The Methods

The participants of this study are 53 first-year Computer Science students who enrolled in the Linear Algebra course during the academic year of 2017/2018. The topic chosen for this study is the Inner Product Spaces (IPS) because it requires the accommodation process and some students make errors and construct a wrong generalization (Junus, 2017). They have learned about Euclidian vectors in R^2 and R^3 during high schools. However, the concept of inner product spaces is new for them. The study is guided by the following questions: (1) what misconceptions do students encounter, and (2) what kind of difficulties do they face in studying the subject?

The learning approach to deliver the Inner Product Spaces is conducted as the following steps.

(a) Online small group discussion. The class is divided into eight groups each of which consisting of five to seven students. Each focus group is given a different set and the binary operations: addition and scalar multiplication. Each group is expected to identify the properties of the set over the arithmetic operations. These properties will lead to the axioms of vector spaces.
(b) Interactive lecturing to guide student define vector spaces and inner product functions including the concept of length, angle, and projection of orthogonal vectors in other vectors.
(c) Online discussion. Students are required to discuss and articulate their understanding of the subjects.
(d) Lab work. Students are asked to construct concept maps using The Kit Build tool (Wunnasri et al., 2018). A concept map consists of concepts (nodes) and links. The nodes/concepts are given, and the students define links to build their concept maps. Each link needs to be named meaningfully according to the characteristics of the relation between the concepts.

3.1. Data Collection

The data collection design is presented in Figure 1.

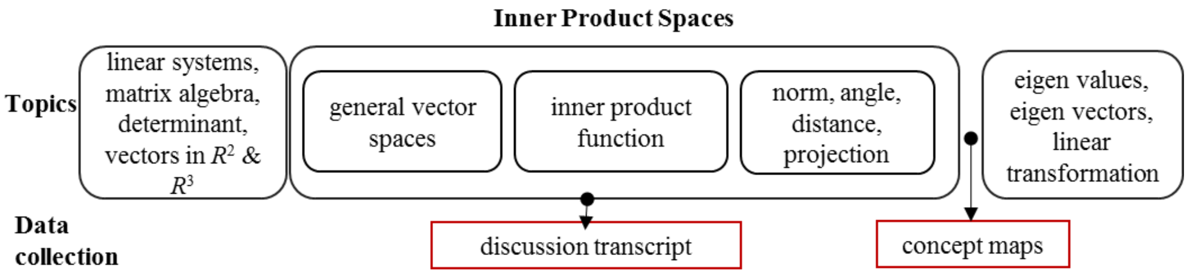

Figure 1. Data Collection Design

3.2. Data Analysis

The discussion transcripts and concept maps are analyzed separately before being compared, integrated, and interpreted. The transcript is coded by two subject experts, who then read the transcript twice. First, they skim the entire transcript and mark the errors. Then, they reread each message and categorize the errors and misconception based on themes (similarity of the causes). To analyze the concept maps, the subject expert and the researcher generate propositions each of which relates two concepts (nodes). For each pair of different concepts, we determined all possible correct links. Two links with synonymous names are regarded as the same link. In addition, propositions: *node A – link from A to B – node B*' and '*node B – link from B to A – node A*' are considered equivalent.

3. The Findings and Discussion

The unit of analysis is a message. The number of student messages to be sampled is 236 units. One message can contain one or more types of misconceptions. Students' misconceptions are categorized as follows.

Table 1

Types of Errors or Misconceptions Found in the Discussion Transcript

Category	Example of excerpt
Incorrect use of letters and symbols to represent objects and operations	"*In a vector space V, for all a, b, and k: $a + b = c \in V$ and $ka \in V$.*" "*V + V is closed.*"
Misconceptions about mathematical objects and their components	"*a function is three in one: domain, codomain, and range.*" "*a vector space is a subspace of an inner product space.*" "*I now understand that a vector is a subset of a vector space.*"
orrect use words when students learn various new concepts that are similar to their pre-knowledge	"*The length of the vector in an inner product space can be calculated using the dot product.*"
Misconception because of the improper understanding of underlying terms/ concepts. Misuse of terms (exchanged) of mathematical objects and their representations	"*I now understand that a vector is a subset of vector spaces.*" "*(2, 5) is a point, so it is a representation of a zero vector positioned there.*" "*... all elements of R ^ 5 can be written as (a, b, c, d, e). It also represents a point in R ^ 5. As we all know, all points are vectors 0. Why? Because we can translate all points into (0, 0, 0, 0, 0) which is the definition of vector 0.* "
Misconception due to differences in focus of view (Calculus and Linear Algebra)	"*y = f(x) = sin x is the vector space consisting of points on the plane so that every point on the curve is a vector in C[0, 1]*"
Misconception due to preconceived notions rooted in their previous knowledge about vectors in R^2 and R^3.	"*In a vector space M consisting of all 2x3 matrices, a matrix A consists of two-row vectors and three column vectors.*" "*We know that R ^ 5 is a vector with 5- coordinates that can lead to 5 different directions*".

The errors may reflect learning difficulties. Students' errors detected form discussion transcript are categorized as the following: incorrect use of terms about elements (of a set) and a set, incorrect notation (errors in writing mathematical symbols), such as scalars and vectors. In addition, they fail to understand the underlying concepts comprehensively. For instance, by not accurately explaining the concepts, misuse of terms, and making wrong generalizations of the concept of inner product spaces. Students have a great deal of difficulties to make abstraction and generalization. Next, we will investigate misconceptions based on students' concept maps. Each student concept map is compared manually to the goal map to identify errors. The comparison was done manually since students may use synonyms or misspell the label of a link.

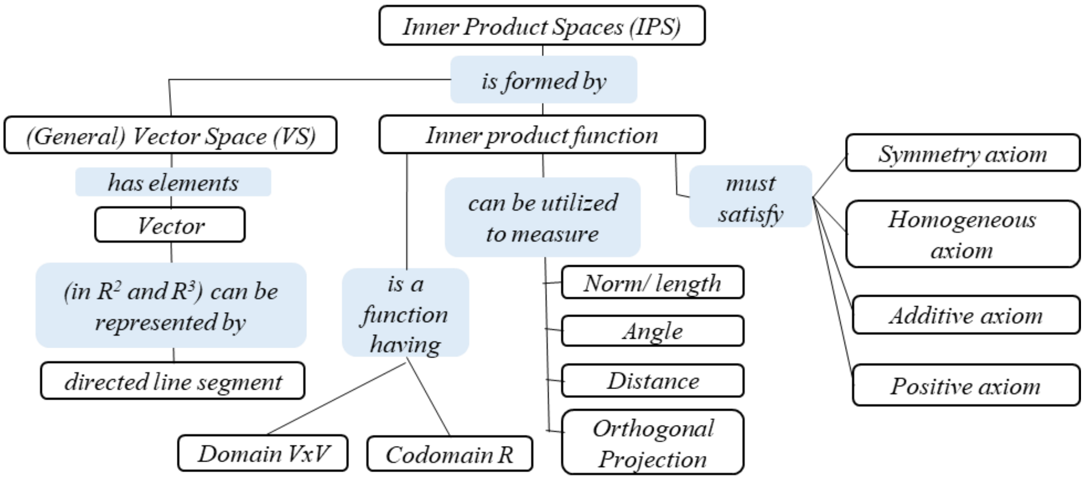

Figure 2. The Experts' Map

The types of misconceptions are categorized based on the incorrect links between pairs of nodes. Referring to the locus of errors and to address the first research question, we focus on the relationships between the following pairs of nodes, as shown in Table 2.

Table 2

Types of Misconception

Node 1	Node 2	Incorrect relation/link	Types of errors/ misconceptions
vector	*directed line segment*	*is defined as*	Confusion about objects and their representations; and how to construct a general vector space
inner product function	*IPS*	no link	Confusion about an entity and its components
inner product function	*NADP measurement*	no link	Students fail to understand that the inner product function is the base to calculate norm, angle, distance, and orthogonal projection.
vector	*VS*	no link	Misconceptions arising from the conflict between new knowledge and pre-knowledge; accommodation is not optimal. Misconceptions about an entity and its components.
inner product function	*SHAP axioms*	no link	Misconceptions about measurements that are relative to the inner product function defined, it is due to the improper understanding of underlying terms/concepts
inner product function	*Domain*: *VxV* *Codomain*: *R*	no link	Misconceptions about an entity and its components, space, and its component
VS	*IPS*	*subspace of*	Misconceptions about an entity its components,
VS	*SHAP axioms*	*has properties*	Misconceptions on how to construct an inner product space

SHAP: *Symmetry*, *Homogeneous*, *Additive*, and *Positive*
NADP: *Norm*, *Angle*, *Distance*, and *Orthogonal Projection*

The students' learning difficulties that may cause the errors are listed below.

- They are confused by the vector definition and vector representations.
- They can explain vectors as elements of a vector space. However, they still keep the definition vectors as entities that can be represented as directed line segments.

- They still perceive the inner product as dot product defined in R^2 or R^3 (failed to generalize)
- They still regard the general vectors Euclidian vectors (fail to generalize)
- They are confused about the inner product space and vector spaces. They fail to construct an Inner Product Space by defining an inner product function in a vector space. They thought that vector space is a subspace of an inner product space.
- They are confused between the nature and the axioms of an algebraic structure
- They do not yet understand the components of an inner product function (that consists of a domain, a codomain, a rule, and axioms).

Based on the types of errors found in the discussion transcripts and concept maps, we categorize misconception as the following.

<u>Misconception 1</u>: Incorrect use of words when students learn various new concepts that are related to one another. Some students use the notions of subspace, vector space, vector space, inner product space incorrectly. In addition, they were also confused with zero vectors and points in a plane. Some students think that (a, b) is a zero vector having an initial point and terminal point at a point $P(a, b)$. This is also the types of misconception related to the representations of mathematical objects.

<u>Misconception 2</u>: Misconceptions arising from the conflict between the new knowledge and the pre-knowledge; wherein the accommodation process is not optimal. Most students already understand that directed line segments represent vectors in R^2 and R^3, yet, some of them still define general vectors as directed line segments. At the same time, they correctly define vectors as the elements of a vector space. This phenomenon shows that based on their perceptions about vectors, there are two groups of students.

- Students who accept the notion that a vector is an element vector space, but they still keep the previous definition that a vector is an entity having both magnitude and direction (can be represented as a directed line segment, the same way as representing the two and three-dimensional Euclidian vectors).
- Students who understand the new concept about vectors and they can describe that directed line segments are geometric representations of vectors in R^2 and R^3. They assert that not all vectors can be represented as directed line segments.

Students have great difficulty in the accommodation process because their previous perception about vector (which can be represented as directed line segments) is profound. They learned such vectors in high school and have them reinforced with the concrete application of physical vectors in their daily life such as velocity, acceleration, and force.

<u>Misconception 3</u>: Misconceptions because of the improper understanding of underlying terms/concepts. Previous students' learning approaches were more focused on procedural knowledge. Therefore, their procedural knowledge is not adequately supported by a conceptual understanding, including the terms and their meanings. The inner product space is a topic that covers numerous terms and basic concepts. Misconceptions occur when they fail to understand the basic concepts. Therefore, they cannot relate among concepts accurately. Additionally, some of them cannot distinguish the between properties (characteristics) and axioms.

<u>Misconception 4</u>: Misconceptions arising from the failure to make generalizations or abstractions. In linear Algebra, students are required to be able to generalize. The design of instruction is prepared to help the students conceptualize the inner product space through examples. They are directed to identify common traits, and then perform abstractions and generations. However, some students are still experiencing difficulties in doing this process due to their previous learning experiences.

Misconceptions that indicate a failure to generalize a concept appear in online discussions and concept maps. For example, a concept map that does not contain a link between the *inner product function* and the *measurements of norm, distance, angle, and orthogonal projection*. Another example is a map without a link between *measurements* and *inner product space*. These show that some students do not understand well that the measurement is relative to the defined inner product. An inner product space can have more than one inner product functions each of which determines the formula for the norm of a vector, distance, angle between two vectors, and orthogonal projection of a vector. Students have difficulty applying what they have learned about inner product functions.

The misconception due to the failure to perform generalization causes failure to construct algebraic structures. For instance, some students claim that vector space is a subspace of inner product space. Based on the finding, activities to avoid such a misconception are proposed, as follows. Students are asked to articulate their understanding of the definition of vector spaces and vectors as vector space elements. Additionally, students are given examples of vector spaces whose elements cannot be presented as directed line segments, such as the vector space consisting of 3x4 matrices.

4. Conclusion

Discussion transcripts can be used to identify errors and misconceptions, such as inaccuracy of the notation writing, concept representation, and definitions of terms. Discussion transcripts can also uncover the misconceptions about relationships between concepts. However, a concept map more clearly identifies the misconceptions associated with inter-concept relations. Both the discussion transcript and the concept map can indicate misconceptions arise from preconception.

Misconceptions should be corrected intentionally to prevent other errors and promote deep learning. Knowing learners' misconceptions enable educators to help them rectify and correct their errors and misconception. Therefore, they have to be corrected. Unfortunately, most often, learners were not aware of the inaccuracy and incompleteness of their mathematical propositions. Therefore, the crucial step is to promote students' awareness of their misconception. They can also help each other by diagnosing others' error. The following are some strategies to remedy misconceptions in this context. First, exposing learners to tasks and situations that trigger them to be aware of their error. Confronting students with their misconceptions make them aware of their misconceptions and how to rectify them. Next, reinforcement and internalization of new knowledge through articulation and practice with various cases and more concrete examples. Provide students with models on how to perform generalizations and abstractions (thinking out loud). Learners also need to be familiarized and trained how to present mathematical definitions and notations in a precise manner.

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