Wong, L.-H. et al. (Eds.) (2013). Proceedings of the 21st International Conference on Computers in Education. Indonesia: Asia-Pacific Society for Computers in Education

# Mathematical Model For Collaborative Learning: Acquiring Hierarchic-Structured Knowledge

**Kohei OGAWA[a], Yasuyuki NAKAMURA[a,*], Koichi YASUTAKE[b], Osamu YAMAKAWA[c]**
[a]*Graduate School of Information Science, Nagoya University, Japan*
[b]*Graduate School of Social Sciences, Hiroshima University, Japan*
[c]*Center for Arts and Sciences,Fukui Prefectural University, Japan*
*nakamura@nagoya-u.jp

**Abstract:** In this paper, time evolutions of students' knowledge level who are engaged in collaborative learning, is simulated using mathematical model. In this model, students try to acquire hierarchic-structured knowledge. It is found that the structure of the collaborative groups formed by the students influence their achievements. Collaborative learning is said to be useful because one can reach the level where one cannot reach with the traditional teaching approach. We have the result that collaborative learning is especially effective when learning the difficult knowledge and we might be able to say our model successfully described the aspect of collaborative learning.

**Keywords:** Collaborative learning, Hierarchic-structured knowledge, Mathematical model approach, Computer simulation

## 1. Introduction

In recent years, there have been theoretical studies of teaching-learning process especially in a collaborative learning. Since Hake reported that the performance of the students can be enhanced using a teaching approach involving collaborative group work, in contrast to the traditional non-interactive lectures (Hake, 1998), the processes of learning and understanding physics and mathematics have become the focus of cognitive research. In order to precede the research, mathematical model of teaching-learning process have been proposed and studied.

There are some mathematical models of teaching-learning process and we classify these models into three categories; differential equation modeling (Pritchard, et al., 2008), Ising spin modeling (Bordogna et al., 2001, 2003, Yau-Yuen, 2006, Yasutake, 2011) and stochastic process modeling (Nitta, 2010). In this study, we adopt Ising spin modeling since the model is most-investigated one among mathematical learning models. In the Bordogna's model, a collaboration results in exchanging student's knowledge with each other. However the knowledge was just a block of what he/she knew in the model. Actually, one knowledge should be based on related basic knowledges, namely hierarchic structure of knowledge. This is similar to hierarchic model of data, information, knowledge, wisdom, known as DIKW Hirarchy (Ackoff, 1989).

In this study, we propose a mathematical model of collaborative learning and we suppose that students exchange hierarchic knowledge with each other. The aim of this study is to clarify how students acquire knowlege and what kind of manner of grouping of students are effective for collaborative learning.

## 2. The model and the simulation method

When we work out a solution to the problem, various types of knowledge are necessary. So we suppose a hierarchic structure of knowledge shown in figure 1. It shows that one needs both knowledge A and B, which are the knowledge that lies under the knowledge D, to learn knowledge D and same thing applies to knowledge E and F. The knowledge level of the *i*th student at time *t* is given by $S_i(t)(0 \leq S_i(t) \leq 1)$, where $S_i(t) = 1$ corresponds to optimum knowledge and $S_i(t) = 0$ corresponds to no knowledge at all. The knowledge level $S_i(t)$ consists of knowledge levels $S_{A,i}(t)$, $S_{B,i}(t)$, ..., $S_{F,i}(t)$.

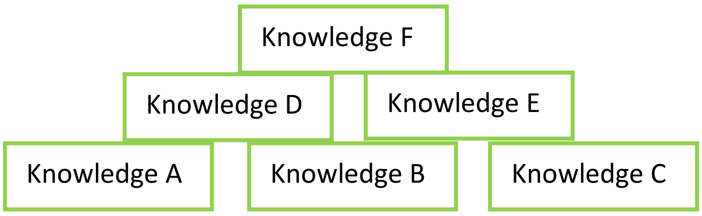



Figure 1. Hierarchic structure of knowledge

A class is divided to some groups and each group consists of *n* students. The cognitive impact of the student-student interaction in a group for knowledge A~C, $C_{X,i}^{SS}(t)$ (A~C applies for X) is given by

$$C_{X,i}^{SS}(t) = \sum_{j=1, j\neq i}^{n} [I_{ij}(t)(1 - A_i(t))\,(S_{X,j}(t) - S_{X,i}(t))], \tag{1}$$

where $I_{ij}(t)$ is the power of influence of *j*th student on *i*th student and $A_i(t)$ is the confidence of the *i*th student. Cognitive impact is expected to be larger when the power of influence is large. The more confidence the student has, the less cognitive impact the student gets, because he/she has the confidence that his/her idea is correct meaning there is less chance to change his mind. The term $(S_{X,j}(t) - S_{X,i}(t))$ shows that the bigger the difference between the *i*th student's knowledge level and the *j*th student's knowledge level, the larger the cognitive impact will be. It also shows that cognitive impact will be negative when the *j*th student's knowledge level is smaller than the *i*th student's knowledge level and positive in opposite case.

We model that the cognitive impact for knowledge D~F, $C_{Y,i}^{SS}(t)$ (D~F applies for Y) depends on the knowledge level of basic (or lower) layer one and the model is described as

$$C_{Y,i}^{SS}(t) = \sum_{j=1, j\neq i}^{n} [I_{ij}(t)(1 - A_i(t))\,(S_{Y,j}(t) - S_{Y,i}(t))(S_{YA,i} + S_{YB,i})/2]. \tag{2}$$

$S_{YA,i}$ and $S_{YB,i}$ is the two knowledge levels that lies under the knowledge level Y. The term $(S_{YA,i} + S_{YB,i})$ means that when your knowledge levels of A and B are small, you have less chance to gain cognitive impact on knowledge D. $I_{ij}(t)$ and $A_i(t)$ is assumed to be given as

$$I_{ij}(t) = I_{ij}^0 |S_j(t) - S_i(t)| \tag{3}$$

and

$$A_i(t) = A_i^0 S_i(t), \tag{4}$$

where $I_{ij}^0$ is the value of influence that *j*th student potentially has on *i*th student. It depends on many factors such as the relationship between the *j*th student and *i*th student, persuasiveness, assertiveness, etc. $A_i^0$ is the value of confidence that *i*th student potentially has. It is provided by the student's characteristic.

The cognitive impact of teacher on *i*th student can be written as

$$C_{X,i}^{TS}(t) = I_{iT}(S_T - S_{X,i}(t)) \tag{5}$$

and

$$C_{Y,i}^{TS}(t) = I_{iT}(S_T - S_{Y,i}(t))(S_{YA,i} + S_{YB,i}), \tag{6}$$

where $S_T$ and $I_{iT}$ are the knowledge level of the teacher and his/her power of influence on *i*th student. $I_{iT}$ is assumed to be given by

$$I_{iT}(t) = I_{iT}^0 |S_T - S_i(t)|, \tag{7}$$

where $I_{iT}^{0}$ is the value of influence that the teacher potentially has on the *i*th student. In the case of $S_i(t) > S_j(t)$, *i*th student will try to teach the *j*th student and when he/she does, what he/she doesn't know becomes clear. So he/she can learn more efficiently from his/her teacher. Therefore, if $S_i(t)$ is larger than $S_i(t)$, $C_i^{TS}(t)$ is amplified as

$$C_i^{TS}(t) = C_i^{TS}(t)(1 + (S_T - S_i(t)), \tag{8}$$

which means that $C_i^{TS}(t)$ becomes larger when the *i*th student has small knowledge level because he/she has lots to learn and has better chance of gaining larger cognitive impact.

The knowledge is assumed to be a dynamic variable influenced by the cognitive impact as Bordogna described. At a given time interval $\Delta t$, the student's knowledge level changes as follows:

i. a knowledge gain of amount $\Delta S_i(t)$, i.e.

$$S_i(t + \Delta t) = S_i(t) + \Delta S_i(t) \text{ with a probability of } P_{i1} = e^{\beta(C_i(t) - \alpha)}/Z \tag{9}$$

or

ii. no knowledge gain, i.e.

$$S_i(t + \Delta t) = S_i(t) \text{ with a probability of } P_{i2} = 1/Z \tag{10}$$

or

iii. a knowledge loss of amount $\Delta S_i(t), i.e.$

$$S_i(t + \Delta t) = S_i(t) - \Delta S_i(t) \text{ with a probability of } P_{i3} = e^{\beta(-C_i(t) - \alpha)}/Z, \tag{11}$$

where Z and $\Delta S_i(t)$ are given by

$$Z = e^{\beta(C_i(t) - \alpha)} + 1 + e^{\beta(-C_i(t) - \alpha)} \tag{12}$$

and

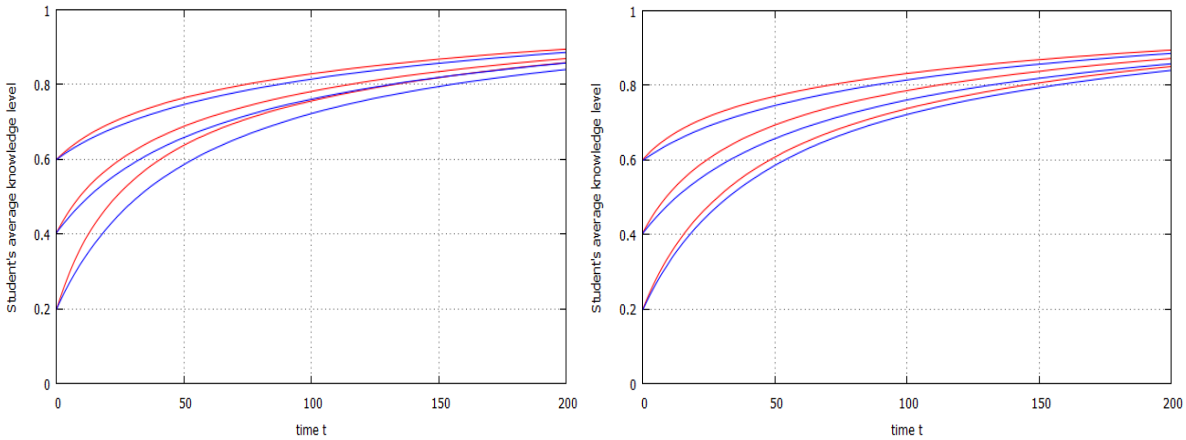
$$\Delta S_i(t) = (1 - | S_i(t)|)/10 \tag{13}$$

in which $C_i(t)$ is the sum of $C_i^{TS}$ and $C_i^{SS}$. $\Delta S_i(t)$ is larger when the student's knowledge level is low and it is smaller when his/her knowledge level is close to the optimum knowledge. This is because the less knowledge one has the more things one has to learn and if one's knowledge level reaches 1, one has nothing to learn and $\Delta S_i(t)$ becomes 0.

It is assumed that $S_T = 1$, $\alpha = 0.3$, $\beta = 2.0$, *i*=1,…,*N* in this paper. *N* is the total number of students in the classroom and $I_{iT}^{0}$ is taken at random in the interval of (0.6,1). Also $I_{iT}^{0}$ and $A_i^{0}$ in (0,1).

## 3. Result and discussion

We take the class size *N*=99 which is divided into 33 groups with 3 students in each group. Students take previous diagnostic test, and are classified into three different sets, namely "high achieving (HA) students" with $\langle S_{HA} \rangle \approx 0.6$, "average achieving (AA) students" with $\langle S_{AA} \rangle \approx 0.4$, and "low achieving (LA) students" with $\langle S_{LA} \rangle \approx 0.2$. Subsequently, three different cases are considered. In Case I, the students only learn from the teacher, which corresponds to the so called traditional teaching approach. In Cases II and III, the students not only learn from the teacher but also engage in collaborative work and interact with other students. However, these two groups are formed differently: in Case II the groups are homogeneous and all the members of each group are selected from same group so that they have same level of knowledge. In Case III the groups are heterogeneous and a member is chosen from each set so that they have different level of knowledge.




(a) (b)

Figure 2.Time evolutions of students' knowledge level in a homogeneous group (a) and in a heterogeneous group (b). Collaborative learning is included in the red curves (Case II in (a) and III in (b)) whereas the blue curves contain no student-student interaction (Case I).

Figures 2 (a) and (b) show the time evolutions of knowledge level for (a) Cases I and II and for (b) Cases I and III. All students' performance is enhanced when they are engaged in collaborative learning (Cases II and III) as compared to the traditional approach (Case I). LA students perform better when the group is homogeneous. On the other hand, HA students and AA students perform better when the group is heterogeneous. This result is contrast to that of Bordogna and Albano (Bordogna et al., 2001, 2003). Their results are as follows: LA students perform much better when they interact with HA students and their achievement was either worse or indistinguishable as compared to the traditional lectures when they interact with LA students. But HA students perform much better when they form homogeneous groups. In figure 2 the knowledge level does not reach 1 but when we have enough simulation time, the knowledge level gets close to 1. We need to check up how long the simulation time is equivalent to actual studying time.

Figure 3 shows the time evolution of students' average knowledge level in a traditional teaching approach (Case I), homogeneous group (Case II) and in a heterogeneous group (Case III). Students perform better when they are engaged in collaborative learning regardless of grouping. They perform slightly better when they are formed in homogeneous groups than formed in heterogeneous group.

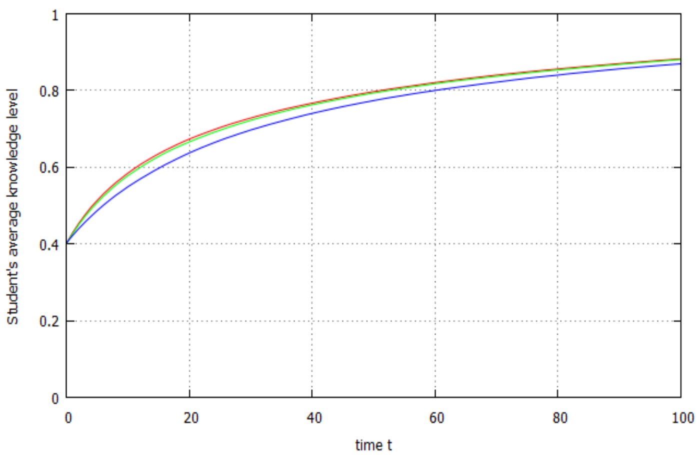



Figure 3. Time evolutions of students' average knowledge level in a traditional teaching approach (blue curve), homogeneous group (red curve) and in a heterogeneous group (green curve).

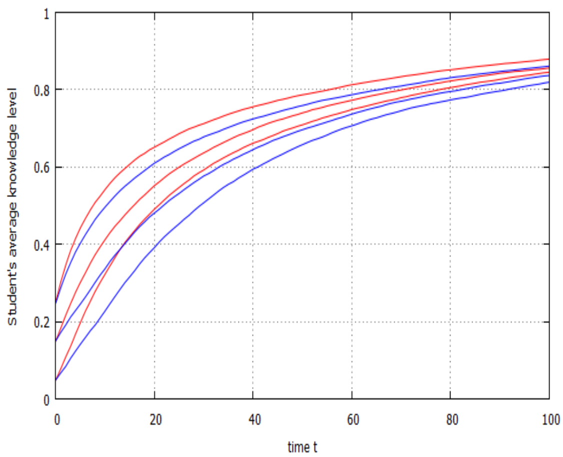
Figures 4 (a) and (b) show the time evolution of knowledge level A, D and F for Cases I and II for low achieving students (a) and for high achieving students. Both results show that collaborative learning is effective for all knowledge but especially when learning difficult knowledge. Collaborative learning is said to be useful because one can reach the level where one cannot reach with the traditional teaching approach. We might be able to say this result is showing the aspect of the collaborative learning.

To show the effectiveness of collaborative learning using quantitative index, let us introduce Hake's actual gain G and show the result in figure 5. Here, gain G is defined as the difference between the primary knowledge level and the knowledge level at time 40 in the simulation. We have chosen time 40 because its result was most close to the result of Hake's. In figure 5, each points corresponds to each student's result. We can see the collaborative learning (red points) is slightly more effective than the traditional learning (green points). The result is qualitatively equivalent to Hake's results and this means that our model might be appropriate mathematical model of learning.

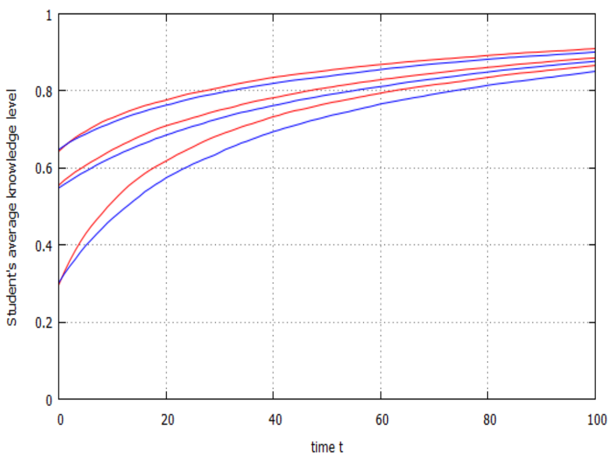
## 4. Conclusion

Mathematical learning model has been formulated to simulate the collaborative learning processes provided by any intelligent tutoring systems. It is found that collaborative learning is effective in both grouping, homogeneous grouping and heterogeneous grouping. The students' average knowledge level was slightly better in the case of forming homogeneous groups than in the case of forming heterogeneous groups. But AA students and HA students perform better in heterogeneous groups so we cannot necessarily say forming homogeneous groups is better than forming heterogeneous groups. Result showed collaborative learning is effective when learning difficult knowledge. This might help teachers to improve their teaching strategies.

It is important to compare the results of the present studies with the results of actually implemented collaborative learning. Then we reformulate and make the more reliable mathematical model and propose a good condition for students to learn. In the procedure, not only qualitative but also quantitative investigation is required. In this paper, we introduced students' gain defined by Hake, which we calculated by the use of the numerical results of the time 0 and 40. In order to further enhance authenticity, it should be investigated how long the simulation time is equivalent to actual studying time. We hope that this type of numerical study will help in developing education.




(a)




(b)

Figure 4. Time evolutions of students' knowledge level A (highest pair of curves), D (middle pair of curves) and F (lowest pair of curves) in a homogeneous group. (a) shows the knowledge level of low achieving students and (b) shows the knowledge level of high achieving students.

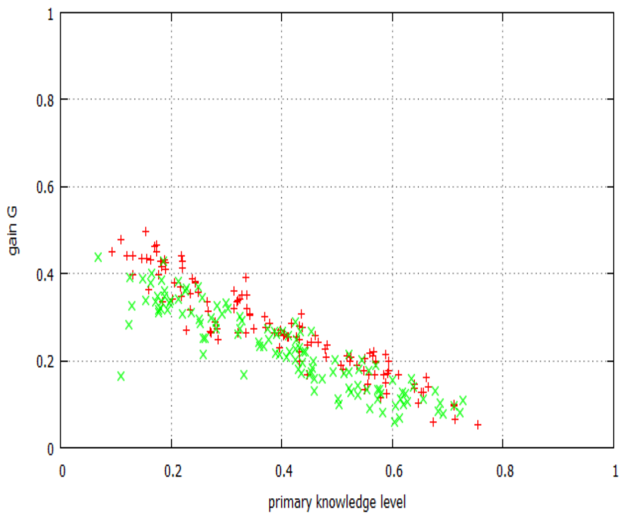



Figure 5. Gain G vs. primary knowledge level. Red points are the students engaged in collaborative learning and green points are the students only engaged in traditional learning.

## Acknowledgements

This work was partially supported by KAKENHI (Grant-in-Aid for Scientific Research) (24650557) from Japan Society for the Promotion of Science (JSPS).

## References

Ackoff, R. L. (1989). From data to wisdom, *Journal of Applied Systems Analysis*, *16*(1), 3-9.

Bordogna, C. M. & Albano, E. V. (2001). Theoretical Description Teaching-Learning Processes: A Multidisciplinary Approach. *Physical Review Letters*, *87*(11), 118701-1-4.

Bordogna, C. M. & Albano, E. V. (2003). Simulation of social processes:application to social learning, *Physica A: Statistical Mechanics and its Applications*, *329*(1-2), 281-286.

Hake, R. R. (1998). Interactive-engagement vs. traditional methods: A six-thousand-student survey of mechanics test data for introduc- tory physics courses, *American Journal of Physics*, *66*(1), 64-74.

Nitta, H. (2010). Mathematical theory of peer-instruction dynamics, *Physical Review Special Topics - Physics Education Research 6*, 020105-1-4.

Pritchard, D. H. & Lee, Y.-J. (2008). Mathematical learning models that depend on prior knowledge and instructional strategies, *Physical Review Special Topics - Physics Education Research*, *4*, 010109-1-8.

Yeung, Y.-Y. (2006). Scientific Modeling of Technology-Mediated Collaborative LearningProcess. In Mizoguchi, R., Dillenbourg, P. & Zhu, Z. (Eds), *Learning by Effective Utilizationof Technologies: Facilitating Intercultual Understanding*, (pp. 249-256). Amsterdam, The Netherlands: IOS Press.

Yasutake, K., Yamakawa, O., Tagawa, T., Sumiya, T. & Inoue, H. (2011). A Simulation Analysis on Learning Effects through Interactions in Networked Communities (*in Japanese*), *Transactions of Japanese Society for Information and Systems in Education*, *28*(1), 50-60.