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A System that Supports Learners' Strategic Thinking for Solving Highschool Mathematics Problems

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Abstract: In high-school mathematics, learners sometimes cannot solve problems well even if they remember formulas. At other times, learners cannot solve similar problems well even if they see a sample answer of the original problem. These cases indicate that *mathematical knowledge*, such as knowledge of the formulas and definitions in high school math textbooks, is insufficient to solve problems. Past studies have shown that it is important to plan which formula should be used in problem solving. The knowledge necessary to determine which formula should be used in problem-solving is called *knowledge of the problem-solving strategy*. In the past researchers on educational support systems, there is no established method for mathematics education support systems to construct a problem-solving process with the knowledge of the problem-solving like a human doing. In this study, we analyze problem-solving process in which learners use *the knowledge of problem-solving strategy* from specific examples. After that, we construct a system with problem-solving ability using *the knowledge of the problem-solving strategy*. Our system can also verbalize problem-solving processes constructed by itself and let learners read the generated commentary. We conducted a preliminary experiment comparing the commentary with sample answers in usual reference books. In this preliminary experiment, we evaluated the correct answer rate of similar questions and the acquisition rate of *the knowledge of problem-solving strategy*. As a result, subjects reading the commentary by the system mark better correct answer rate of similar problems, and they can reproduce *the knowledge of problem-solving strategy* better than ones reading reference books.

Keywords: problem-solving, high school mathematics, knowledge of problem-solving strategy

1. Introduction

In high school mathematics, sometimes learners cannot solve problems well even if they remember formulas. At other times, even if they see the sample answer of the original problem, they cannot solve a similar problem well. These cases indicate that *mathematical knowledge*, such as formulas and definitions in high school math textbooks, is not enough to solve problems. Then, what kind of abilities and knowledge other than *mathematical knowledge* is necessary? There are researchers who say that it requires various abilities. According to Garderen (2006), spatial visualization ability is important for solving math problems. Based on the level of understanding capabilities of each student, Ramdhani (2017) reports that students who have a high, moderate, or low self-efficacy master the indicators of mathematical understanding. Daniel and Michèle (2007) report that a wide variety of abilities such as reading comprehension and information processing is required for solving math problems. Tarzimah and Thamby (2010) report that various skills such as number fact, arithmetic, information, language, and visual–spatial skills are necessary in solving mathematics problems. They also report that the most important of these skills is the information skill, which is the expertise in connecting information to a concept, operation, and experience or transferring information and transforming problems into a mathematical sentence.

As in these studies, problem solving in mathematics requires a variety of abilities. These abilities affect the learner's thinking, but it is difficult to directly observe it. Therefore, we focused on the learner's problem-solving process. The problem-solving process is either written in answers or implicit. We thought to analyze not only the part written in the answer but also the implicit part. Polya's model, Mayer's model, and Kintsch's model are well-known examples of the models of the mathematical problem-solving process. Polya (1962) proposed that the mathematical problem-solving process consists of four stages: "understanding the problem," "devising a plan," "carrying out the plan," and "looking back." Kintsch and Greeno (1985) proposed that the mathematical problemsolving process can be classified into both the "problem understanding process" and "problem-solving process." Mayer (1992) proposed that the "problem understanding process" consists of "conversion" and "integration," and the "problem-solving process" consists of "planning" and "execution." The focus of each of these models is to understand the problem and plan for problem-solving. Chinnappan and Lawson (1996) report that the students in the experimental group who are encouraged to plan and learn have higher problem-solving ability than those in the control group who only learned using examples. From these previous studies, not only mathematical knowledge but also *the knowledge of the problem-solving strategy*, such as planning what formulas to use and when and how to use them, are important for problem solving.

There are LEAP, MOLE, and SEEK2 as educational systems that focus on the problemsolving process. Mitchell (1985) analyzed expert problem-solving processes at LEAP and LEAP learns the used knowledge as new rules. Politakis (1984) compared the problem-solving process of a system to that of an expert at MOLE and fixed a bug. Ginsberg (1985) automated some of the bug fixes of MOLE at SEEK2. These systems are excellent for acquiring knowledge, but not for evaluating learners' answers. Brown (1978) proposed an educational system that can point out why the answer is wrong by modeling the learner's error as a bug model. This system is good at pointing out errors in learners' answers, but it is not suitable for explaining why the correct answer used a particular formula.

There is no established method for mathematics educational systems to construct a problemsolving process with *the knowledge of the problem-solving* like a human doing. We thought that if the system could solve the problem using *the knowledge of the problem-solving strategy*, the system can point out specifically which part of the learner's answer is wrong and why. It can also explain why the correct answer used a particular formula. Therefore, we analyze problem-solving process in which learners use *the knowledge of problem-solving strategy* from specific examples. In this paper, we construct the system with the ability to solve problems by itself using *the knowledge of the problemsolving strategy*. In addition, our system can also verbalize problem-solving processes constructed by itself and let learners read the generated commentary. We also conducted a simple preliminary experiment to measure the learning effect when learners read the commentary output by the system. As a result, it was suggested that learners who read the commentary output by the system readily improve their ability to solve similar problems and acquire the knowledge of the problem-solving strategies better than those who read the commentary in usual reference books.

2. Basic discussion

2.1 Problem domain

In this paper, we select trigonometric functions in high school mathematics as a problem domain. That is because the field of trigonometric functions requires *the knowledge of the problem-solving strategy*. We made a case study regarding how learners build a problem-solving process based on 30 trigonometric exercises from a high school math reference book. From the problem-solving process, we extracted the *knowledge of the problem-solving strategy* and *mathematical knowledge*.

2.2 problem-solving process

As a result of the examination, we find that the problem-solving process is configured as shown in Figure 1. Each step in this problem-solving process employs a single strategy. Particularly, each step is comprised of the following flow. In this paper, we refer to expressions that are set as the initial conditions of the problem and the goal of the problem or expressions that are inferred through reasoning as "known expressions."

<1> *Known expression* and manipulation are written. (2.2.1 “manipulation”)
<2> Conditions for manipulation are written. (2.2.2 “Condition of applying the strategy”)
<3> Benefits of manipulation are written. (2.2.3 “benefits of manipulation”)

There are two types of methods for the problem-solving step, forward reasoning and backward reasoning. Forward reasoning is the right arrow in Figure 1, and backward reasoning is the left arrow in Figure 1. The *known expression* obtained by forward reasoning is called “*forward known expression.*” The *known expression* obtained by backward reasoning is called “*backward known expression.*”

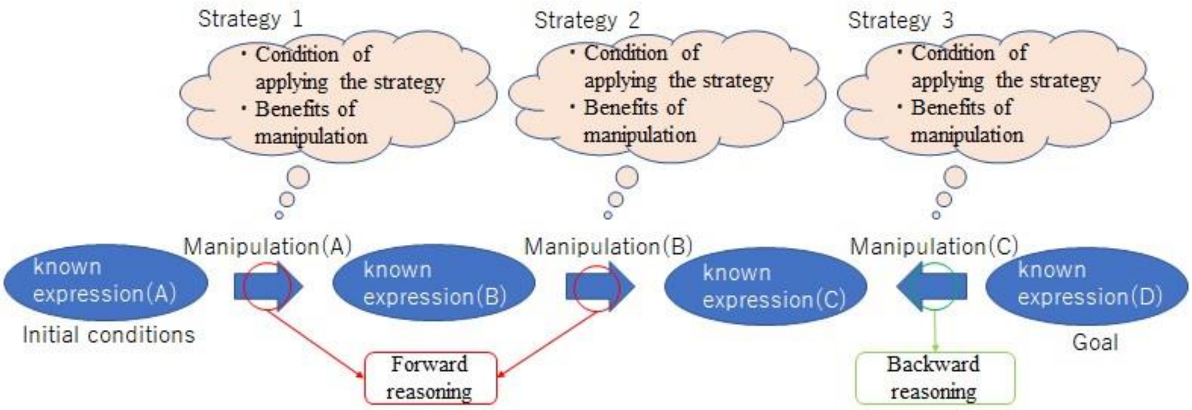

Figure 1. Problem-solving process

2.2.1 Manipulation

In Figure 1, manipulation is the conversion of a formula that occurs when any *knowledge of a problem-solving strategy* is applied. *Manipulation* serves as a bridge between the *mathematical knowledge* and *problem-solving strategy knowledge*. For example, assume that there is a strategy that is based on a *mathematical knowledge* “double-angle formula (Sin[2*x] = 2*Cos[x]*Sin[x]).” Then, the *manipulation* of the strategy is “applying Sin[2*x] = 2*Cos[x]*Sin[x] to the target formula.”

2.2.2 Condition of applying the strategy

The “condition of applying the strategy” in Figure 1 is a condition that determines whether a certain strategy is available or not. For example, to use the double angle formula (Sin[2x] = 2Cos[x]Sin[x]), "Sin [2*x]" must exist in the formula of the *known expression*. To rationalize the denominator, a root must exist in the denominator of the *known expression*. In the case of forward reasoning, the “condition of applying the strategy” is a condition targeting initial conditions and the *known expression of forward*. In the case of backward reasoning, the “condition of applying the strategy” is a condition targeting the goal and the *known expression of backward*.

2.2.3 Benefits of manipulation

The “benefits of manipulation” (BoM) show what kind of benefits can be obtained by actually applying a strategy. The BoM is used to determine how applying a strategy helps to solve problemsolving. In the BoM, there is no distinction between forward reasoning and backward reasoning.

The BoM can be classified into (A) “benefits of a single strategy” and (B) “benefits of multiple strategies.”

(A) *benefits of a single strategy*: benefits of a single strategy are benefits that help problem-solving. *Benefits of a single strategy* are provided when applying a single strategy. For example, using the double-angle formula gives the benefits of “declination coefficient can be 1.”

(B) *benefits of multiple strategies*: *benefits of multiple strategies* are provided when predicting the next strategy to apply. The *benefits of multiple strategies* also apply multiple strategies. The *benefits of multiple strategies* are the same as the *benefits of a single strategy*, and applying multiple strategies can provide benefits that help problem-solving. Additionally, *the benefits of multiple strategies* also have the benefits of looking to a next step, such as “Formula 2 can be transformed into a form to which

Strategy 2 can be applied," as shown in Figure 2. For example, when a form "Sin[x] + Cos[x]" exists in a formula of a *known expression*, the formula can be converted to the form "Sin[x]^2 + 2*Cos[x]*Sin[x] + Cos[x]^2" by performing the operation of raising both sides to the second power. Hence, you can obtain the benefits of the "formula: Sin[x]^2 + Cos[x]^2 = 1 can be applied."

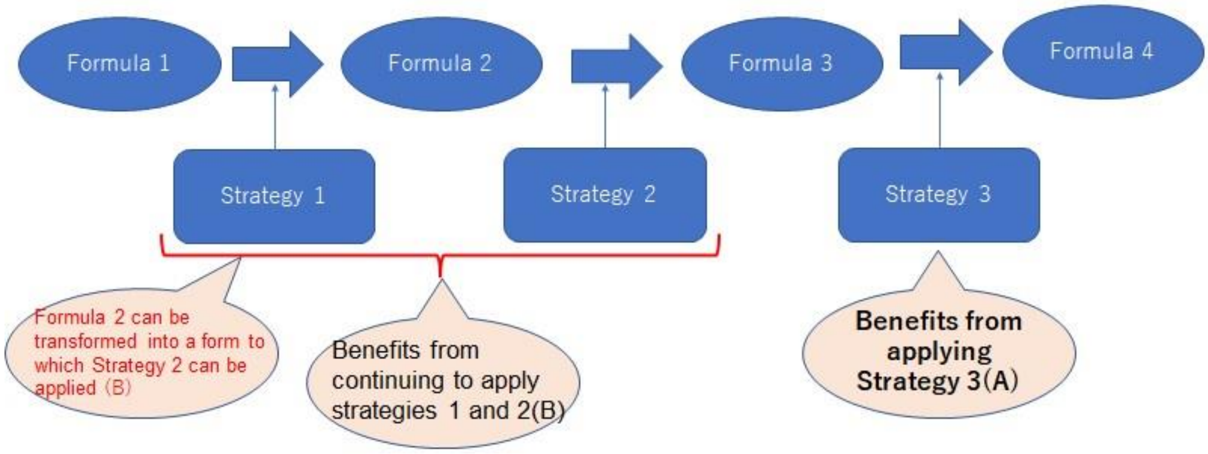

Figure 2. Classification of BoM (A), (B)

Alternatively, the "benefits of manipulation" can be classified into (α), (β), and (γ) from the viewpoint of factors that cause the benefits.

(α) benefits always provided: benefits always provided are benefits that are provided regardless of the form of the *known expression* to which the strategy is applied.

(β) benefits that are provided if conditions are satisfied: *benefits that are provided if the conditions are satisfied* are benefits that are provided when a formula to which the strategy is applied satisfied certain conditions for each benefit.

(γ) benefits that are provided as a result: unlike (α) and (β), *benefits that are provided as a result* are not provided when a specific strategy is applied. *Benefits that are provided as a result* are benefits that are provided when the result is a specific pattern no matter which strategy is applied.

As an example of (α)(β)(γ), consider the case of using the double-angle formula for a formula of the *known expression*. Using the double-angle formula, a benefit of the "declination coefficient can be 1" is provided as (α). If the declination of the *known expression* is [x] except for Sin [2*x], problem solvers can obtain the benefit of the "declination in the formula can be unified" as (β). If the result of the manipulation can be reduced, problem solvers can obtain the benefit of "formulas can be reduced and simplified" as (γ).

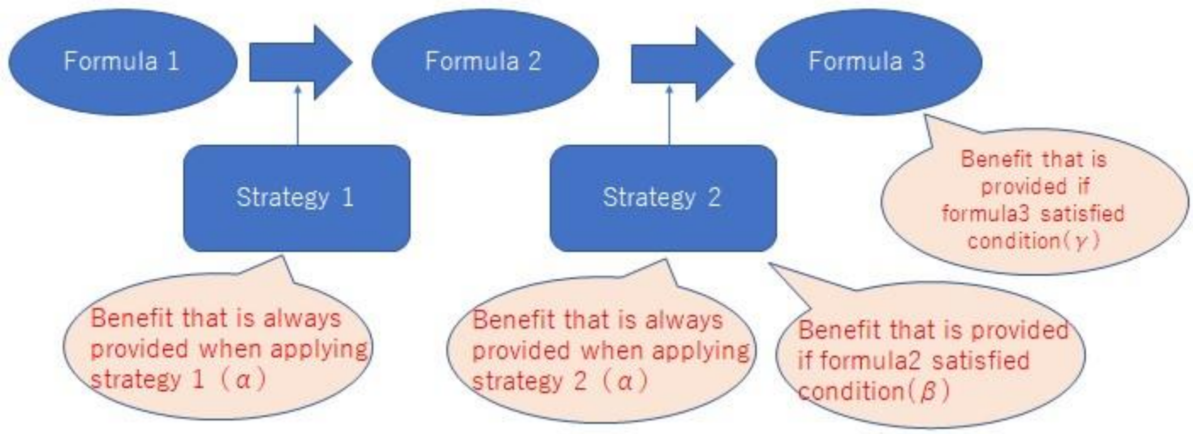

Figure 3. Classification of BoM (α), (β) and (γ)

2.3 Knowledge of problem-solving strategy

As discussed thus far, the problem-solving process has *manipulation*, the *condition of applying the strategy*, and the *benefits of manipulation*, as shown in Figure 1. As a result, the *knowledge of the problem-solving strategy* must include the ability for the *condition of applying the strategy and* *manipulate as a factor*. Additionally, the *benefits of manipulation* must be considered because (γ) *benefits that are provided as a result* are provided with reference only to the result of applying the strategy and are used to guide the overall problem-solving. Therefore, the *knowledge of problemsolving strategy* was then composed of the *manipulation*, *condition of applying the strategy*, and *benefits of manipulation* other than (γ) *benefits that are provided as a result*.

2.4 Problem-solving algorithm

We were able to sort out the flow of the learner's building problem-solving process. Now, we build a problem-solving algorithm that constructs the problem-solving process according to the flow shown in Figure 1. In preparation for building the algorithm, we define the "inference distance." The *inference distance* measures a degree of coincidence between the *forward known expression* and *backward known expression*. The *inference distance* is generally judged by BoM, degree of a formula, number of terms and so on. However, the inference distance is currently judged only by the equality of *the expressions*. When the *inference distance* becomes zero, *forward known expression* and *backward known expression* are the same formula. Then the problem-solving is successful. The flow of the algorithm is as follows:

<1>Ask the learner to enter initial conditions and goals (Save each information as a *known expression*).

<2>Check if each strategy satisfies the *condition of applying the strategy* for the currently declared *known expression*, and then, pick up the strategy. However, if the same strategy was applied to the same information in the past, it does not pick up. If no strategy satisfied the *condition of applying the strategy*, we fail to build a problem-solving process.

<3>Apply all the strategies picked up in <2> to the *known expression* that satisfies the conditions.

<4>Generate the *known expression* based on the result of applying the strategy.

<5>Measure all the inference distances between the forward and backward *known expression* pairs. If the inference distance is zero, the problem-solving process is built successfully. If the inference distance is not zero, return to <2>.

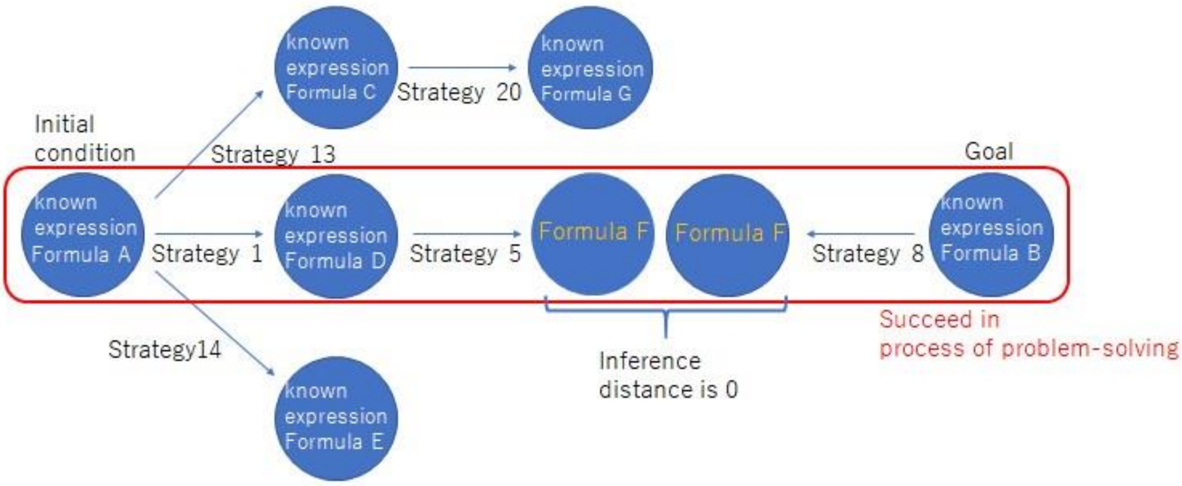

Figure 4. An example of problem-solving process

Figure 4 shows the flow of building a problem-solving process according to the algorithm. In Figure 4, the initial condition is Formula A, and the goal is Formula B. (<1>) The system first looks for a strategy that satisfies the *condition of applying the strategy* for Formula A or Formula B. (<2>) As a result, strategy 13, strategy 14, and strategy 1 apply to Formula A, and strategy 8 applies to Formula B. (<3>) By applying all applicable strategies, the system can obtain new Formulas C, D, and E by forward reasoning and formula F by backward reasoning. (<4>) The system measures all the inference distances between the forward and backward *known expression* pairs. (<5>) As a result, the inference distance is not zero. Thus, the system looks for a strategy that satisfies the *condition of applying the strategy* for Formulas C, D, E, and F again. (<2>) It finds that strategy 20 applies to formula C and strategy 5 applies to Formula D. (<3>) By applying all of these, the system can obtain new Formulas G and F by forward reasoning. (<4>) The system measures all the *inference distances* between the forward and backward *known expression* pairs again. <5> As a result, the *inference distance* between Formula F of the *forward known expression* and formula F of the *backward known expression* is zero. Therefore, the system succeeds in building a problem-solving process.

Through this algorithm, the system attempts to build a process based on exhaustive searches using a strategy that satisfies the *condition of applying the strategy*. This search is not very heuristic. Therefore, the system may not be able to explain to the learner why the strategy was selected from strategies that satisfied the condition. Therefore, the system cannot explain why a strategy was chosen. However, it can explain the benefits of using that strategy. Because each strategy used in the problem-solving process built by the system has BoM. We believe that learners can understand effective strategies for problems and learn guidelines for problem solving.

2.5 Generate a commentary

The system can output a commentary that emphasizes the *knowledge of the problem-solving strategy* (Figure 5). We believe that the learners can become aware of the *knowledge of the problem-solving strategy* by reading the commentary. The commentary is generated by preparing a template and applying the process of problem-solving maintained by the system to it, but the details are omitted in this paper.

In the commentary, each element of *the knowledge of the problem-solving strategy* is shown with emphasis. Step 1 represents the content of one *knowledge of the problem-solving strategy*. In addition, in the output concerning the *benefits of manipulation*, the notation method is changed for each classification of benefits. (α) The benefits always provided, as shown in <1> of Figure 5, can be written as [standard tactics] to learn what guidelines should be applied to the strategy used. (β) The *benefits that are provided if the conditions are satisfied*, as shown in <2> of Figure 5, can be notated as [the benefits of the manipulation in this problem] to indicate the usefulness of the strategy in a particular situation. (γ) The *benefits that are provided as a result* are denoted as [the result of applying the strategy] and can convey to the learner a guideline for the overall problem-solving.

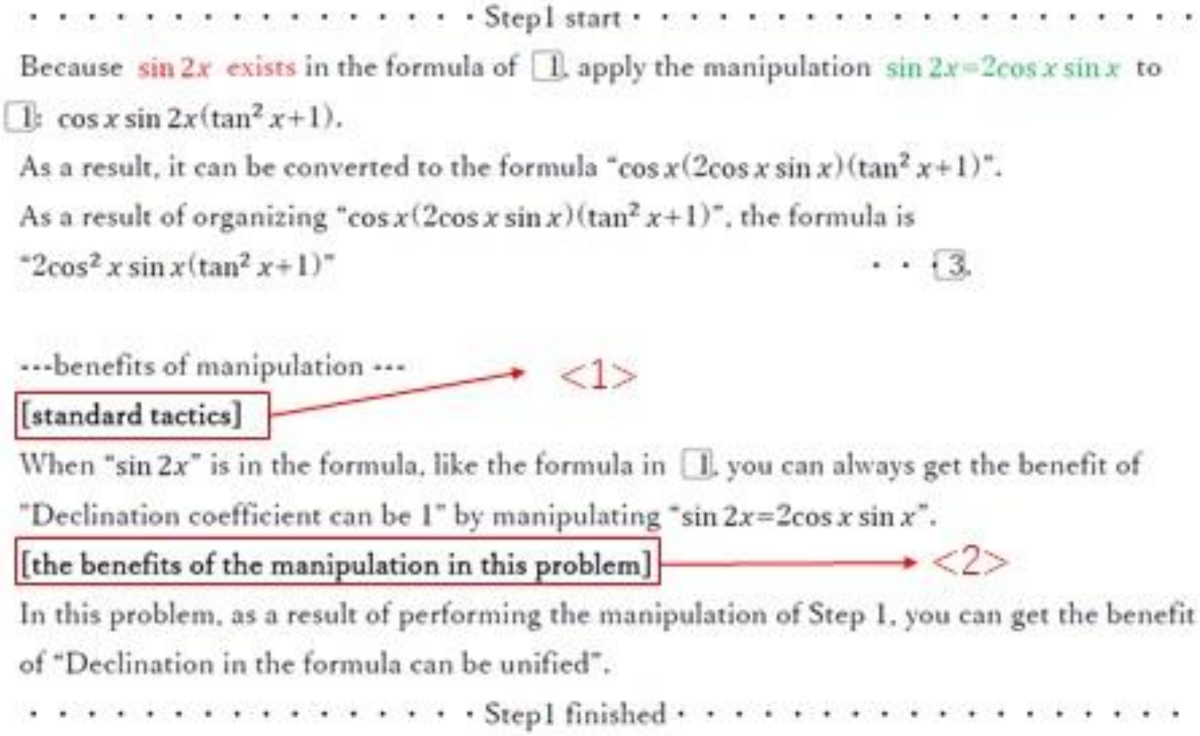

Figure 5. The commentary output by the system

3 Implementation

3.1 System architecture

The system architecture is shown in Figure 6. A learner enters the initial conditions and goals in the system's input/output UI. The system uses the problem solver to build a problem-solving process with the strategy database as a reference. Then, the system uses the commentary generator to generate a commentary.

When the system receives input on the initial conditions and goals of the problem, it outputs the commentary as an HTML file of the constructed problem-solving process. Formulas in the commentary are output by MathML, a markup language. This allows learners to read the formulas, such as fractions and exponents, in a form that is familiar to them and reduces the burden of reading explanations.

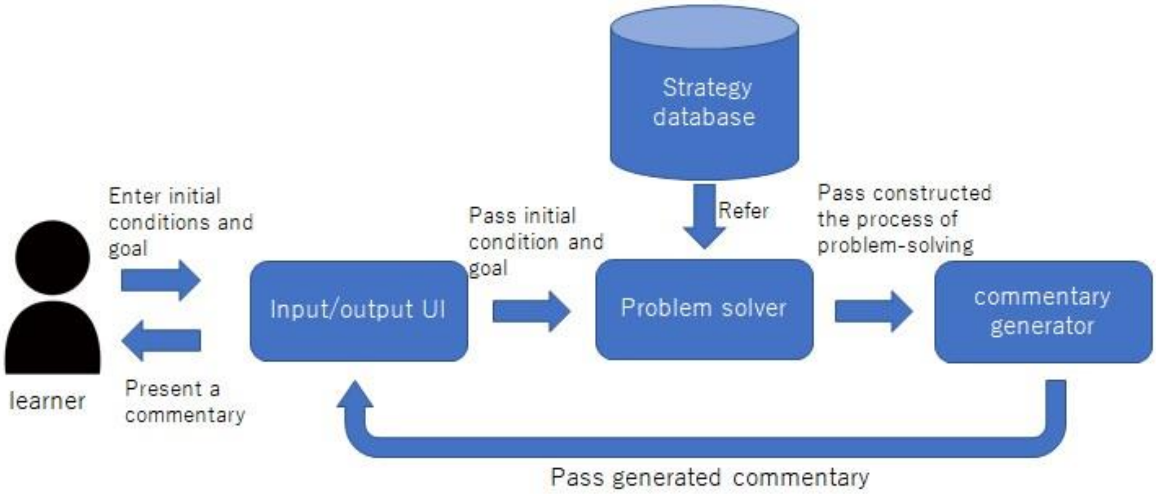

Figure 6. System architecture

3.2 Database of problem-solving strategy

Currently, the system has approximately 30 strategies. The system can build a problem-solving process within the range of the 30 strategies combined. With the current system, the construction of a problem-solving process can cover approximately 60% of the trigonometric proof problems in high school mathematics in a usual reference book sold in Japan.

3.3 Process of mathematical deformation

Mathematical deformation is necessary for the system to apply strategies and organize the information it holds. When the system applies strategies to formulas, such as adding, subtracting, modifying, and applying formulas, it uses Wolfram Research's Mathematica (Wolfram Research 2022). Mathematica is a mathematical deformation processing system. Mathematica is also used to convert the form of a formula to MathML when generating commentary.

4 Preliminary experimental evaluation

An evaluation experiment was conducted to determine if the system would be useful for learners to acquire *the knowledge of the problem-solving strategy* and increased problem-solving ability for similar problems. This experiment is a preliminary experiment conducted with a small number of subjects.

4.1 Experimental hypothesis

The experimental hypothesis of this experiment are as follows:

Hypothesis 1:By reading the commentary output by the system, learners can acquire the *problem- solving strategy knowledge* better than learning without the system.
Hypothesis 2: By reading the commentary output by the system, learners are able to solve similar problems better than learning without the system.
Hypothesis 3: Learners who understands *the knowledge of the problem-solving strategy* are able to solve similar problems better than learners who do not understand the knowledge.

4.2 Method of the experiment

The subjects of this experiment are six university students. Subjects are grouped into Group A and Group B, each of which includes three subjects. Two problems are used in this experiment: Problem 1 and Problem 2. These problems should have the same level of difficulty as much as possible. For that reason, we chose these problems having the same number of steps. We regard these problems as having approximately the same difficulty. At first, an explanation of what is the knowledge of the problem-solving strategy is given to subjects. It explains that *the knowledge of the problem-solving strategy* consists of *manipulation*, the *condition of applying the strategy*, and the BoM with concrete examples, as shown by four slides. Then, the following flow is conducted.

<1> Groups A and B solve Problem 1. Next, Group A reads the commentary generated by the system. Group B reads the commentary in the usual reference book.
<2> Subject writes the *knowledge of the problem-solving strategy* learned in <1>.
<3> Groups A and B solve two similar problems of Problem 1. (Similar problem is the problem in which the same strategy as that of Problem 1 is used.)
<4> Subject writes the knowledge of the problem-solving strategy learned in <3>.

Next, the similar flow as <1> to <4> is executed with Problem 2.
In these steps, the roles of Group A and Group B are interchanged. Specifically, Group A reads the commentary in the reference book, and Group B reads the commentary generated by the system.
Finally, the subjects answer the following questionnaire:
(Q1) Was it possible to be aware of the *knowledge of the problem-solving strategy* by using the system?
(Q2) Did you find it easier to solve similar problems by using the system? (Q3)
Please write your opinion on this system freely.

4.3 Result

The results of the preliminary experimental evaluation are presented in Table 1, Table 2, Table 3, and Figure 7. In Table 1 and Table 3, the subject's answers in step <4> are graded by the following flow:
・If *the condition of applying the strategy* is collected, score plus 1.

・If the *manipulation* is collected, score plus 1

・If the *benefits of manipulation* is collected, score plus 1.

As a result, the answer is graded on a scale of one to three.

Hypothesis 1: As a result, the average score of the knowledge of the problem-solving strategy for subjects who solved the problem reading the commentary of system was 2.67 points, as shown in Table 1. The average score of the knowledge of the problem-solving strategy for subjects who solved the problem with reading the commentary of reference book was 0.75 points, as shown in Table 2. Therefore, subjects can learn knowledge of the problem-solving strategy better if you read the commentary generated by the system. As a result of the questionnaire (Q1), all the subjects answered: "I was very conscious" or "I was a little conscious," as shown in Figure 7. Therefore, the experimental hypothesis 1 was found to be supported.

Hypothesis 2: The correct answer rate of similar problems for subjects reading the commentary of system was 91.6%, as shown in Table 2. The correct answer rate of similar problems for subjects reading the commentary of reference book was 75.0%, as shown in Table 2. Therefore, subjects who read the commentary of system has a higher percentage of correct answers for similar questions than subjects who read the commentary of reference book. As a result of the questionnaire (Q2), 83.3% of the subjects answered, "Thanks to the system, it was very easy to solve similar problems" or "Thanks to the system, it was a little easy to tackle similar problems," as shown in Figure 7. Therefore, the experimental hypothesis 2 was found to be supported.

Hypothesis 3: The correct answer rate of similar problems for the subjects who understand *the knowledge of problem-solving strategy* was 80%, as shown in Table 3. The corrects answer rate for the subjects who did not understand *the knowledge of the problem-solving strategy* was 71%, as shown in Table 3. Therefore, the experimental hypothesis 3 was found to be supported.

Table 1. *Knowledge of problem-solving strategy each group* on the step<4>

	Group A	Group B
Commentary of System	**3.0**	**2.3**
Reference book	**0.8**	**0.7**

Table 2. *Correct answer rate for similar Score in problems in each group on the step<3>*

	Group A	Group B	Average
Commentary of System	**100%**	**83%**	**91.6%**
Reference book	**67%**	**83%**	**75.0%**

Table 3. *Difference of correct answer rate caused by understanding the knowledge of problem-solving strategy*

		Group A	Group B	Total correct answer rate
Subjects who understand the strategy（score in the step <4> is 3）	**Correct answers**	**8**	**0**	**80% (4/5)**
	Incorrect answers	**0**	**2**	
Subjects who don't understand the strategy (score in the step <4> is less than 3)	**Correct answers**	**2**	**8**	**71% (5/7)**
	Incorrect answers	**2**	**2**	

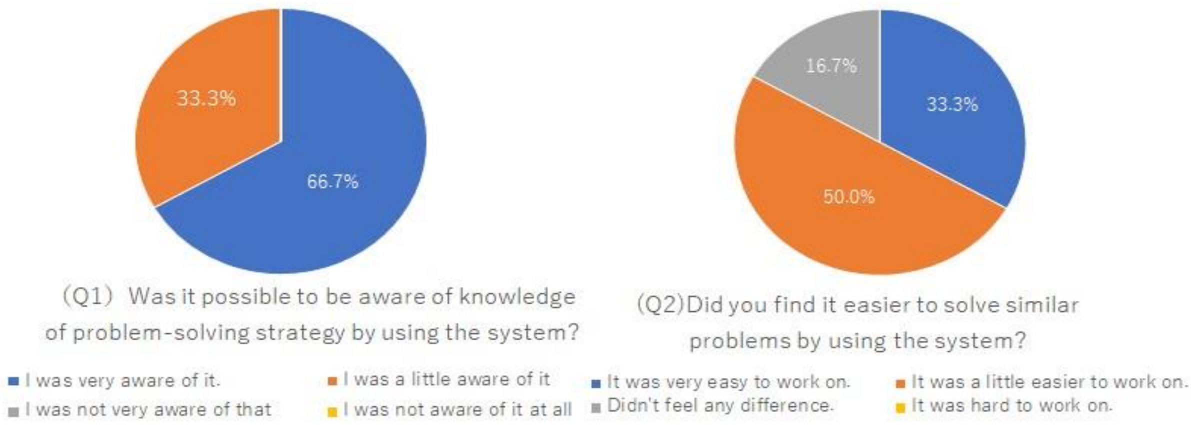

Figure 7. Answers to questionnaire(Q1) and (Q2)

5. Conclusion

In this research, we construct a system with an ability to solve problems by itself using *the knowledge of the problem-solving strategy*. After that, we built a function to verbalize the constructed process and output it as a commentary. Results of the preliminary experimental evaluation proved that the system helps learners acquire *the knowledge of the problem-solving strategy* and their ability to solve similar problems. The system is useful for learners who face mathematics-specific difficulties and can show them how to think about those difficulties. We believe this research can propose one of the useful methods in the educational support system for mathematics.

In the future, the system will be able to evaluate the validity of the answers and provide specific advice for their answers. If it can evaluate the validity of the learner's answer, it will be able to detect alternative answers and provide more flexible learning support. To provide specific advice to the learner's response, it will consider the presence of disadvantages rather than the presence of benefits.

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