A Visualization App on Proving Geometric Concepts

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Abstract: This paper discusses the description, design and pedagogical basis of a mathematical app called *Two Column Proof*, which provides students a framework for writing proofs of geometric statements. The app focuses on proving concepts on: properties of parallelograms, conditions that determine when a quadrilateral is a parallelogram, properties on trapezoids and kites for high school mathematics. It employs visual representations for students to understand the statement or reason of each given line in the proof, strengthening their logical and mathematical knowledge needed in the proof construction.

Keywords: Mathematical app, two column proof, STEM, mathematical proofs

1. Introduction

The abilities to carry out analytical and critical thinking are necessary for students to succeed in STEM education. In the STEM fields, analytical and critical thinking skills are used to find patterns in data, determine the meaning of these patterns, and then use the data to support a claim (Adams, 2017). Mathematics educators have been recommending increasing the role of proofs in the mathematics education of students at all levels (Stylianou et al, 2009). Moreover, it has been suggested that the role of proofs be opportunities for students to observe patterns, make conjectures and provide justification, not just merely exercises in logical thinking (De Villers, 2003). Geometry is one of the subject areas where mathematical argumentation and proving is taught, developing students' deductive reasoning and critical thinking (Kunimune, Fujita, & Jones, 2010). The various geometric facts and theorems may not have much use to students in all aspects of STEM education, but the critical thinking capability developed through the rigor of mathematical proofs would certainly make them more successful problem solvers. Moreover the concept of proof is used frequently in STEM education, where students need to write reasons and justifications based on evidence in a logical manner and organized manner.

This paper discusses a mathematical application (app) called *Two Column Proof* (TCP), designed and created to support the proving of geometry concepts. Zbiek (2003) emphasizes that the design and appropriate use of technology is essential in developing mathematical intuition, understanding math concepts and proving assumptions. TCP is one of the mathematical resources developed under government-funded projects (De Las Peñas et al., 2022, 2023) with one of the primary objectives to strengthen mathematical learning competencies in students. In engagement in schools under these projects, one of the greatest challenges of teachers is the teaching of geometric proofs. Students have difficulties completing proofs. There is not much student engagement with the topic, and the students have difficulty learning as well as enjoying the learning process.

This paper presents the design and pedagogical basis for the TCP app, where the visualization of geometric representations is integrated into the app with the purpose of helping students understand the proof construction process; allowing them to have a better understanding of the arguments to use to support statements and claims. Employing visualization technological tools stimulates interest in learning and developing cognitive interest (Andreev et al., 2021). Visualization representations and the role of visualization play an important role in geometry learning (Žakelj & Klančar, 2022). The use of visualization in proofs was found to be effective in discovery learning models (Kristiyajati & Wijaya, 2019). Visualization is a powerful tool to explore mathematical problems, give meaning to mathematical concepts and find relationships between them (Rösken & Rolka, 2006).

The TCP app was designed to run primarily on mobile devices (Android) and desktops (Windows). The Android version can run on devices with Android 5.1 until 13, 2 GB RAM, and approximately 120 MB available storage. The Windows version can run on 32- or 64-bit systems with Windows 7 or later, 1 GHz processor or higher, 4 GB RAM, and approximately 300 MB available storage. In the past two years of the pandemic, the app has been useful in support of the blended learning modality in schools called for by the Philippine Department of Education (DepEd). The app can be used with minimal supervision, and is apt for remote learning. An interested reader is invited to visit the Mathplus website at https://mathplusresources.wordpress.com/ for access to the app and other mathematical resources developed by the authors.

2. Scientific Framework and Pedagogical Basis

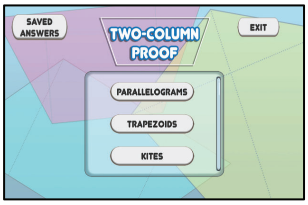
The design of the *Two Column Proof* app was grounded on the use of visualization and imagery in mathematics, which has drawn the attention of experts in the field of psychology of mathematics education (Presmeg, 2006). Drawing on the extensive body of work on mathematical visualization, Rivera (2011) presented a strong rationale for a visually-oriented mathematics curriculum. An important aspect of this curriculum is the use of diagrams. Pantzaria et al. (2009) found that diagrams were not universally helpful. In fact, the presence of diagrams constrained the range of strategies employed by students when solving problems. However, presenting a problem together with a diagram significantly enabled students with initial difficulties to solve problems successfully. Rivera argued that the main issue is not whether to present diagrams or not, but more importantly, how diagrams are used by the students. Drawing on the idea of diagrams as effective visual scaffolds provided that students use them meaningfully, the app presents a diagram integrated into a feedback mechanism representing a target theorem. More specific details on how the app facilitates effective use of diagrams towards the proof of a theorem are discussed in the next section.

3. The *Two Column Proof* (TCP) App

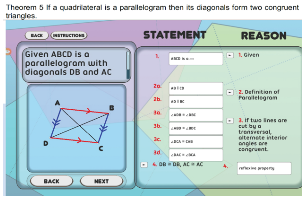
3.1 App Description

A *two column proof* is a structured way of coming up with a proof of a mathematical result, where in one column, a statement is written, and in the second column a corresponding justification and reason is given for the statement. The two-column proof remains the most common proof format in many secondary school textbooks. The first encounter of students in proving geometry concepts is usually through the two-column proof.

The *Two Column Proof* (TCP) app was designed primarily to help students in writing a geometric proof. The application is divided into three main topics: parallelograms, trapezoids and kites (Figure 1(a)). The app addresses the following learning competencies in the Philippine high school curriculum: i) "determine the properties of a parallelogram"; ii) "determine the conditions that make a quadrilateral a parallelogram"; and "prove theorems on trapezoids and kites" (Department of Education [DepEd], 2020). The first part of the parallelogram sub-menu gives statements pertaining to properties of parallelograms; the second part pertains to the converse of these statements, where a property of a quadrilateral is given and one has to prove the quadrilateral is a parallelogram. For the trapezoid and the kite sub-menus, statements on properties of trapezoids and kites are given respectively. While writing a proof in the app, the student clicks the "Next" button to proceed to the next Statement/Reason pair. A line of the proof appears, and the student types the answer in the blank provided. To visualize a statement/phrase, the right arrow button is clicked (Figure 1(b)).



(a)



(b)

Figure 1. (a) Main menu; (b) "Statement" and Reason" columns with the Next/Back buttons

3.2 Design

The key element considered in the design of the app is *visualization*. Žakelj & Klančar (2022) defined visualization as the creation, application and reflection of diverse representations. In the process of visualization, the student creates, identifies or shapes visual representations and applies them to solve problems. In the design of TCP*,* geometric concepts can be visualized with constructed representations that are dynamic in nature. In particular TCP provides visual feedback for a particular line in the proof. Each line in the proof is detailed, in order to depict visually the necessary connections between each one. The visualization element is intended to guide the student in writing the mathematical statements, providing the

Table 1. *Visualization Representations in the Two Column Proof App*

Task	Visualization	Example from the app
To write mathematical statements in the "Statement" column where the justification provided in the "Reason" column is the definition of a geometric figure, such as a parallelogram, isosceles trapezoid or kite.	Relevant parts of the geometric figure are highlighted such as pairs of sides or angles. Color flashes are provided as visual hints.	In proving "The diagonal of a kite is an angle bisector of a pair of opposite angles," in the "Reason" column the definition of kite is given. To visualize the properties that distinguish a kite from other quadrilaterals, the animation involves the first pair of sides AB and AD (repeated flashes in blue) (Figure 2(a)); followed by the second pair of sides, BC and DC (repeated flashes in red) (Figure 2(b)). Congruence of sides is hinted at by highlighting the congruence symbols.
To write mathematical statements in the "Statement" column where the justification provided in the "Reason" column is a geometric result a student has been previously taught.	An animation that will illustrate the theorem is shown.	In proving "Opposite sides of a parallelogram are congruent," given in the "Reason" column is "If two parallel sides of a parallelogram are cut by a transversal then alternate interior angles are congruent". The animation first shows one pair of parallel sides AD and BC (Figure 2(c)), followed by a transversal BD (Figure 2(d)) then the pair of alternate interior angles ADB, CBD (Figure 2(e)). The process is repeated for the other pair of parallel sides AB, DC, the transversal BD, then the pair of alternate interior angles ABD, CDB (Figure 2(f)).
To provide the correct triangle congruence theorem or postulate in the "Reason" column that justifies the given statement in the "Statement" column.	Animation is provided on the corresponding parts of both triangles; then the two triangles are highlighted.	For the Angle-Side-Angle (ASA) congruence used in proving a property of a parallelogram "Opposite sides are congruent," an animation is provided in the following sequence: a pair of angles ADB then CBD from two different triangles (Figure 2(g)), an included side (Figure 2(h)), a second pair of angles ABD then CBD of the same two triangles (Figure 2(i)). Then the two triangles are highlighted, one after the other, first triangle ABD followed by triangle CDB (Figure 2(j)).

correct reasons to the given statements or understanding the logical order of the proof. We outline in Table 1 examples of visualization or visual representations used in the app.

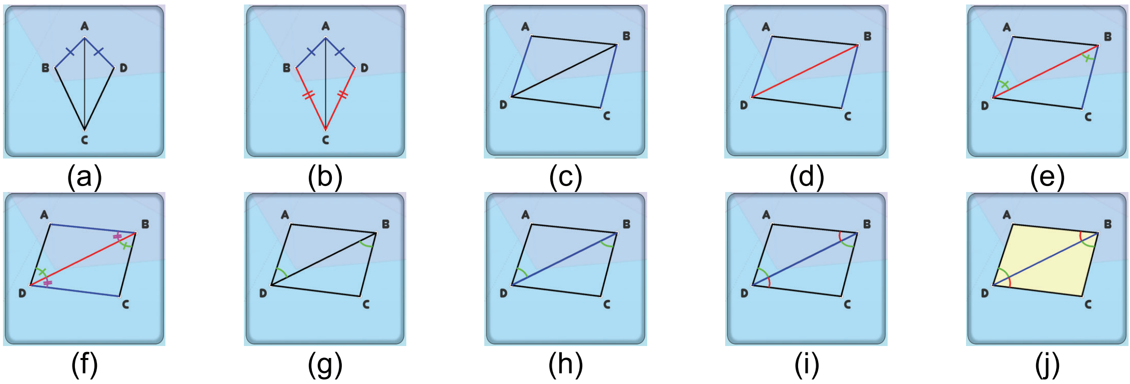


Figure 2. Animation for (a)-(b) definition of a kite; (c)-(f) "If two parallel sides of a parallelogram are cut by a transversal then alternate interior angles are congruent"; and (g)-(j) Angle-Side-Angle (ASA) congruence.

One of the features of TCP is that a student may save the answers to the lines of the proof. An answer key is provided by the app once he/she has finished drawing up a complete proof and saved the work; The student can rework the proof even after viewing the answer key and save a revised version of the proof.

4. Integration and Use of *Two Column Proof* (TCP) App

TCP can be used as a tool in teaching based on the RAT (Replacement, Amplification, Transformation) framework (Hughes, Thomas, & Scharber, 2006) where the app may replace the conventional method of teaching proofs without affecting the content or the learning objectives associated with proofs or the act of proving. It is common for teachers to write two columns on the board in the classroom to start the process of proving geometric statements. This practice is now replaced by the interface in the app. The task of completing the proof of a theorem remains the same but the static method of presenting them is now changed to the colorful and dynamic animation of geometric definitions and properties. Such features of the app can be used by the teacher as a scaffold to help students understand relationships between parts of the geometric figure.

Additionally, the app's integration can go beyond simple replacement. The app can amplify learning. The app offers a visualization and animation of angles and sides of geometric shapes that can communicate more effectively and clearly the connections between the angles and sides. The correct properties and definitions to employ as justifications in the proof are subsequently identified by the students with the help of these visual clues.

Last but not least, technology may transform teaching strategies and the learning process. The app may encourage teachers to shift from teacher-centric or teacher-controlled instruction to student-centered instruction. As student engagement is enhanced by the feedback and save-answer mechanisms, the learning process may change. To increase their exposure and practice, students can simply repeat and rework the proof. By doing this, they will be better able to comprehend the logical sequence of a proof. They can also compare kinds of proofs and proof strategies such as that of a theorem and its converse by repeatedly going through the lines of the proofs and their corresponding visual images. Also, the app can be utilized as a tool for group work in which students express their ideas and experiences while using it. This kind of activity also encourages collaboration among students and fosters a supportive learning atmosphere. Further, the app can be utilized in a face-to-face or distance learning set-up.

In terms of the advantage and disadvantage of using the Android and desktop versions of TCP: the desktop version has the advantage of letting the students type the proofs into the app with more precision and at times, more speed as the desktop device uses a mouse and a keyboard. The animations would also be noticeably clearer as compared to portable devices with smaller screens. On the other hand, the primary advantage of using TCP in a mobile device is its portability.

In the implementation of the government funded projects (De Las Peñas et al., 2022, 2023) we observed the use of TCP in a Philippine high school in school years 2021-2022 (offsite because of the pandemic) and 2022-2023 (onsite) for selected Grade 9 classes. One of the ways the teachers use the app in class is when introducing the lesson in proving a geometric theorem or when discussing an assigned proving homework. The students found the animations from the app useful during synchronous classes via google classroom or in the classroom when the two column proof was taught for the first time. The app was also helpful for asynchronous classes that followed. The teachers were assured of a way they could check on how the students practiced proving, and they could assign several problems where the students could verify the proofs. A common feedback from the students is that the animations, when carried out repeatedly, were able to guide them in writing down sentences of the proof. Moreover, it was a big help to rely on the feedback of the app at the time when they were not with their teachers.

A study was conducted to assess the effectiveness of the app in the learning of Grade 9 students in a public high school in Quezon City, Philippines during the school year 2022-2023. Two pairs of control-experimental classes were given a pretest and a posttest on the competencies: (i) proves theorems on the different kinds of parallelograms (rectangle, rhombus, square), (ii) proves theorems on trapezoids and kites. The first pair's control class (38 students), where the app was not introduced and only traditional instruction was applied, recorded a mean gain score of 30.94%. On the other hand, the experimental class (40 students), where the app was used in synchronous lessons and in homework, got a mean gain score of 44.90%. The second pair of control and experimental classes recorded a wider gap between mean gain scores: 25.73% for the control (25 students) and 79.29% for the experimental (27 students). These results suggest the potential of the app to help students improve in the mentioned competencies.

5. Conclusion and Future Direction

The *Two Column Proof* app was designed as a technological tool to aid students in learning the process of proving geometric statements. Utilizing visualization features, the app shows definitions or geometric statements in a step-by-step animated format, actively involving students in an entire proving task. While the app focuses on specific learning competencies on parallelograms, kites and trapezoids it can also benefit students in other grades who require guidance in constructing geometric proofs. Studies have shown that using visualization tools helps pupils better understand geometric ideas and properties. In the Two-Column Proof app, students are assisted in developing mathematical arguments and constructing proofs in a clear and coherent manner. This support is provided through the app's visual components, which highlight the relationships between geometric objects such as lines, segments, angles, triangles, and quadrilaterals.

As a next step, a more detailed analysis is being carried out on the effectiveness of the app on student learning and competencies with geometry proofs. There are also plans to extend the app to include proofs of other geometric theorems. Adding more animations and interactive elements can help improve further the learning experience of students.

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